The curve shortening flow in the metric-affine plane

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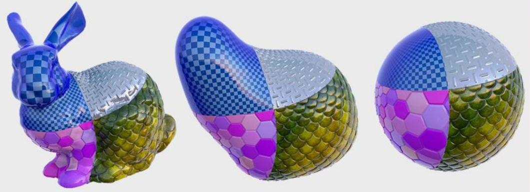
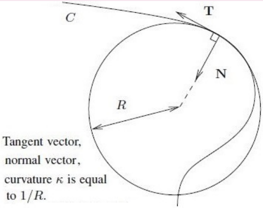


The 1-dim. **mean curvature flow** is called the **curve shortening flow** (CSF). It is the negative L^2-gradient flow of the length of the interface, used in modeling the dynamics of melting solids. The CSF deals with a family of closed curves γ in the plane $\mathbb{R}^2$ with a **Euclidean metric** $g = \langle\cdot\,,\cdot\rangle$ and the **Levi-Civita connection** ∇, satisfying the initial value problem (for $t \ge 0$):

$$\partial\gamma/\partial t = kN, \quad \gamma|_{\,t=0} = \gamma_0. \tag{1}$$

Here, k is the curvature of γ with respect the **unit inner normal vector** N, and γ_0 is an embedded plane curve, see [1, 2, 3, 4].

Recall that the curvature of a convex plane curve is positive. The CSF given by (1) is invariant under translations and rotations. Prehistory: **W.W. Mullins (1956)**.



Prehistory: CSF

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Two-Dimensional Motion of Idealized Grain Boundaries

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To represent ideal grain boundary motion in two dimensions, a rule of motion of plane curves is considered whereby any given point of a curve moves toward its center of curvature with a speed that is proportional to the curvature. A general theorem is deduced concerning the change of area enclosed by such a curve. Three families of curves are found that obey the curvature rule of motion while undergoing the shape preserving transformations of uniform magnification, translation, and rotation respectively. Pieces of these curves represent the steady shapes of idealized grain boundaries under certain symmetrical conditions.

I. INTRODUCTION

It has been shown by Beck[1] that the grain boundaries of a recrystallized metal, when annealed, migrate toward their centers of curvature. The concomitant reduction in the area of the boundaries, all having positive free energies when referred to an equivalent amount of crystal, provides the driving force for this motion. Thus, it is easily shown[2] that a boundary of mean curvature K and free energy per unit area σ is urged toward its nearest center of curvature with a pressure given by $p=K\sigma$. Such pressures and the motions they produce are now recognized to be responsible for normal grain growth.[3]

Smoluchowski[4] and Turnbull[5] have shown that a pressure p, of the type discussed, produces an unbalance in the fluxes of atoms crossing a boundary. This, in turn, causes the boundary to move in the direction favored by the pressure with a speed S given by $S=pM=K\sigma M$, where $M=Ae^{-q/kT}$ is the speed per unit

of a curved film. This drives gas through the film causing it to move toward its center of curvature in a manner similar to that of grain boundaries. There is, however, this important difference between the two cases: within each cell of a soap froth, the possibility of a rapid mass flow of air maintains a uniform pressure which in turn causes each film to have a constant mean curvature; within a metal grain there is no possibility of a rapid mass flow and its associated uniformity of pressure so that the motion of any portion of a boundary is governed by local conditions only. Thus the problem of grain boundary motion, according to the curvature rule, is a problem in differential geometry. We will confine the discussion to the two-dimensional case of plane curves which correspond to grain boundaries in sheets.

II. THE CURVATURE RULE AND THE AREA THEOREM

Consider an arbitrary curve[7] given by $r(\theta,t)$ where r

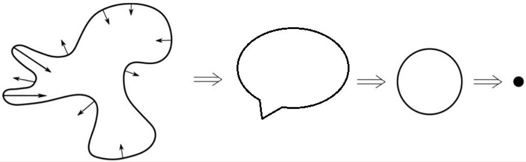
The curve shortening flow

The next theorem describes this flow of convex curves.

Theorem 1 (M. Gage and R.S. Hamilton [2])

a) Under the CSF (1), a convex closed curve in the Euclidean plane smoothly shrinks to a point in finite time (i.e., $t \le \omega < \infty$).
b) Rescaling in order to keep the length constant, the flow converges exponentially fast to a circle in C^∞.

This theorem and further result by M.A. Grayson, [3] (that the flow moves any closed embedded in the Euclidean plane curve in a finite time to a convex curve) have many generalizations and applications in natural and computer sciences.



The **anisotropic curvature flow**, K.-S. Chou and Xi-P. Zhu [1], for closed convex curves in a Euclidean plane generalizes the CSF:

$$\partial\gamma/\partial t = (\Phi(\theta)k + \Psi(\theta))\,N, \quad \gamma|_{\,t=0} = \gamma_0. \tag{2}$$

Here $\Phi > 0$ and Ψ are 2π-periodic functions of the normal to $\gamma(\cdot\,,t)$ and normal angle θ.
Anisotropy of the flow (2), studied when $\Psi > 0$, is indispensable in dealing with phase transition, crystal growth, frame propagation, chemical reaction, and mathematical biology. On the other hand, (2) is a particular case of the flow in a Euclidean plane $\mathbb{R}^2(x_1, x_2)$,

$$\partial\gamma/\partial t = F(\gamma, \theta, k)N, \quad \gamma|_{\,t=0} = \gamma_0,$$

where $\gamma = (\gamma^1, \gamma^2)$ and $F = F(x_1, x_2, x_3, x_4)$ is a given function in $\mathbb{R}^4$, 2π-periodic in $\theta = x_3$.

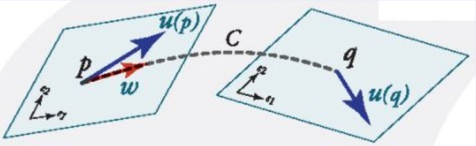
The contorsion tensor

Many results have appeared in the differential geometry of a manifold with an affine connection $\bar\nabla$ (the parallel transport).

The difference $\mathfrak{T} = \bar\nabla - \nabla$

is a (1,2)-tensor,

called **contorsion tensor**.



Two interesting particular cases of $\bar\nabla$ (and $\mathfrak{T}$) are as follows.

1) **Metric compatible connection**: $\bar\nabla g = 0$, i.e.,
$\langle\mathfrak{T}(X,Y),Z\rangle = -\langle\mathfrak{T}(X,Z),Y\rangle$. Einstein-Cartan relativity.

2) **Statistical connection**: $\bar\nabla$ is torsionless and the rank 3 tensor $\bar\nabla g$ is symmetric in all its entries, i.e., $\langle\mathfrak{T}(X,Y),Z\rangle$ is symmetric. Probability and statistics as well as information geometry.

There are no results about the CSF in the metric-affine geometry.

References

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[5] V. Rovenski, **The Curve Shortening Flow in the Metric-Affine Plane**. Mathematics 2020, 8(5), 701; https://doi.org/10.3390/math8050701.

The **metric-affine plane** is a two-dimensional real vector space $\mathbb{R}^2$ endowed with a Euclidean metric g and an affine connection $\bar\nabla$.

To study the CSF for convex curves in the metric-affine plane $(\mathbb{R}^2, g, \bar\nabla)$, we replace (1) by the following initial value problem:

$$\partial\gamma/\partial t = \bar k\, N, \quad \gamma|_{\,t=0} = \gamma_0, \tag{3}$$

where $\bar k$ is the $\bar\nabla$-curvature of γ and γ_0 is a closed convex curve.

Put

$$k_0 := \min\{k(x) : \, x \in \gamma_0\} > 0.$$

Let $\{e_1, e_2\}$ be the orthonormal frame in $(\mathbb{R}^2, g, \bar\nabla)$.
We assume the following

Condition 1: $\mathfrak{T}$ is ∇-parallel, i.e., has constant components $\mathfrak{T}^k_{ij} = \langle \mathfrak{T}(e_i, e_j),\, e_k\rangle$, and its norm $\|\mathfrak{T}\| = c > 0$.

Let $\gamma : S^1 \to \mathbb{R}^2$ be a closed curve in the metric-affine plane with the arclength parameter s. Then $T = \partial\gamma/\partial s$ is the unit vector tangent to γ. In this case,

$$\bar{k} = \langle \bar{\nabla}_T\, T, N \rangle$$

is the curvature of γ with respect to $\bar{\nabla}$, and we obtain

$$\bar{k} = k + \Psi, \tag{4}$$

where Ψ is the following function on γ:

$$\Psi = \langle \mathfrak{T}(T, T), N \rangle. \tag{5}$$

Thus, our flow (3) is the particular case of (2) when $\Phi(\theta) \equiv 1$, but $\Psi(\theta)$ may change its sign. By **Condition 1**, we find

$$|\,\Psi| \le c. \tag{6}$$

Convergence of CSF in a finite time to a point

Our result generalizes Theorem 1(a) and states that a "sufficiently convex" closed curve shrinks to a point under new flow (2), and is asymptotic to a shrinking circle solution of the classical CSF.

Theorem 2 ([5])

Let γ_0 be a closed convex curve in the metric-affine plane with the following condition:

$$k_0 > 2\,c.$$

Then (3) has a unique solution $\gamma(\cdot, t)$, it exists at a finite time interval $[0, \omega)$, and $\gamma(\cdot, t)$ converges as $t \uparrow \omega$ to a point.
Moreover, if $k_0 > 3\,c$ then

$$\omega \le \frac{A(\gamma_0)}{2\pi} \cdot \frac{k_0 - 2c}{k_0 - 3c},$$

where $A(\gamma_0)$ is the area enclosed by γ_0.

The method of [1] to the normalized flow of (2) in the contracting case still works without the positivity of Ψ. Using this and Theorem 2, we obtain the result, generalizing Theorem 1(b).

Theorem 3 ([5])

Consider the normalized curves

$$\tilde\gamma(\cdot, t) = (2(\omega - t))^{1/2}\gamma(\cdot, t),$$

see $\gamma(\cdot, t)$ *in Theorem 2, and introduce a new time variable*

$$\tau = -(1/2)\log(1 - t/\omega) \in [0, \infty).$$

Then $\tilde\gamma(\cdot, \tau)$ *converge to the unit circle smoothly as* $\tau \to \infty$.

Let θ be the **normal angle** for a convex curve $\gamma : S^1 \to \mathbb{R}^2(x^1, x^2)$, i.e., $\cos\theta = -\langle N, e_1\rangle$ and $\sin\theta = -\langle N, e_2\rangle$. Hence,

$$N = -[\cos\theta,\ \sin\theta], \quad T = [-\sin\theta,\ \cos\theta].$$

Lemma 4

The function Ψ *given in* (5) *has the following view:*

$$\Psi = a_{30}\sin^3\theta + a_{03}\cos^3\theta + a_{12}\sin\theta + a_{21}\cos\theta, \tag{7}$$

where

$$a_{12} = \mathfrak{T}^2_{12} + \mathfrak{T}^2_{21} - \mathfrak{T}^1_{11}, \qquad a_{21} = \mathfrak{T}^1_{12} + \mathfrak{T}^1_{21} - \mathfrak{T}^2_{22},$$
$$a_{03} = \mathfrak{T}^2_{22} - \mathfrak{T}^1_{22} - \mathfrak{T}^1_{12} - \mathfrak{T}^1_{21}, \quad a_{30} = \mathfrak{T}^1_{11} - \mathfrak{T}^2_{11} - \mathfrak{T}^2_{12} - \mathfrak{T}^2_{21}.$$

The **Frenet–Serret formulas** $\bar\nabla_T\,T=\bar k\,N$ and $\bar\nabla_T\,N=-\bar k\,T$:

$$\bar\nabla_T\,T=k\,N+\mathfrak{T}(T,T),\quad \bar\nabla_T\,N=-k\,T+\mathfrak{T}(T,N). \tag{8}$$

hold for any curve γ if and only if

$$\langle\mathfrak{T}(T,T),N\rangle=-\langle\mathfrak{T}(T,N),T\rangle,\quad \langle\mathfrak{T}(T,N),N\rangle=0=\langle\mathfrak{T}(T,T),T\rangle.$$

In this case, the following symmetries are valid:

$$\begin{gathered}\mathfrak{T}^1_{12}=-\mathfrak{T}^2_{11},\quad \mathfrak{T}^1_{21}=0=\mathfrak{T}^2_{22},\quad \mathfrak{T}^2_{21}=-\mathfrak{T}^1_{22},\quad \mathfrak{T}^2_{12}=0=\mathfrak{T}^1_{11},\\ a_{12}=-\mathfrak{T}^1_{22},\quad a_{21}=-\mathfrak{T}^2_{11},\quad a_{03}=-a_{30}=\mathfrak{T}^2_{11}-\mathfrak{T}^1_{22},\end{gathered}$$

and the formula

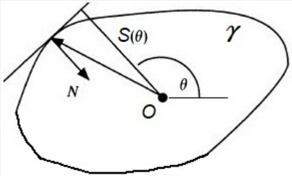
$$\Psi=a_{30}(\sin 3\theta-\cos 3\theta)+a_{12}\sin\theta+a_{21}\cos\theta.$$

The support function

The **support function** S of a convex curve γ is given by

$$S(\theta) = \langle \gamma(\theta), -N \rangle = \gamma^1(\theta)\cos\theta + \gamma^2(\theta)\sin\theta.$$

A circle of radius ρ has $S(\theta) \equiv \rho$.
Since $\langle\, \partial\gamma/\partial\theta, N \rangle = 0$, then
$S_\theta(\theta) = -\gamma^1(\theta)\sin\theta + \gamma^2(\theta)\cos\theta$,
and γ can be represented by S and θ,



$$\gamma^1 = S\cos\theta - S_\theta\sin\theta, \quad \gamma^2 = S\sin\theta + S_\theta\cos\theta. \qquad (9)$$

This yields the following known formula for the curvature of $\gamma(\theta)$:

$$k = (S_{\theta\theta} + S)^{-1}. \qquad (10)$$

Then, according to (4) and (10),

$$\bar{k} = (S_{\theta\theta} + S)^{-1} + \Psi.$$

Local existence and uniqueness of a solution to CSF

Let $\widehat{\gamma}(u, t) : S^1 \times [0, t_1) \to \mathbb{R}^2$ be a family of closed curves satisfying (3). We will use the normal angle θ to parameterize each curve: $\gamma(\theta, t) = \widehat{\gamma}(u(\theta, t), t)$.

Proposition 1

The support function $S(\cdot\,, t) = \langle \gamma(\cdot\,, t), -N \rangle$ of $\gamma(\cdot\,, t)$ satisfies the following PDE:

$$\partial S/\partial t = -(S''_{\theta\theta} + S)^{-1} - \Psi. \tag{11}$$

By the theory of parabolic equations we have the following.

Proposition 2 (Local existence and uniqueness)

Let γ_0 be a convex closed curve in the metric-affine plane. Then there exists a unique family of convex closed curves $\gamma(\cdot, t)$, $t \in [0, t_0)$ with $t_0 > 0$, and $\gamma(\cdot, 0) = \gamma_0$ satisfying (3).

Let $[0,\omega)$ be the maximal time interval for the solution $\gamma(\cdot,t)$ of (3) in the metric-affine plane.

Proposition 3

Let the curvature of γ_0 obey condition

$$k_0 > 2\,c\,. \tag{12}$$

Then the solution $\gamma(\cdot,t)$ of (3) remains convex on $[0,\omega)$ and its curvature has a positive lower bound k_0-2c for all $t\in[0,\omega)$.

Proof. Apply the **maximum principle** to a parabolic equation

$$\partial\bar k/\partial t = k^2(\bar k''_{\theta\theta}+\bar k).$$

Maximum principle. Let $u(x,t)$ be continuous on $S^1\times[0,T]$. Assume $u\le 0$ on $S^1\times\{0\}$. If u is C^2 in x, C^1 in t and satisfies

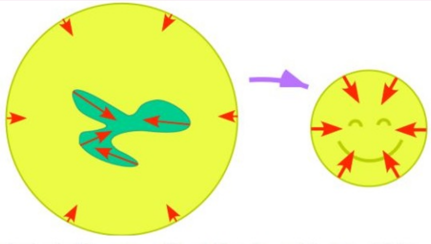
$$\partial u/\partial t \le u''_{xx} + b(x,t)\,u'_x + c\,u$$

with b bounded and $c\in\mathbb{R}$, then $u\le 0$.

Proposition 4

Let convex closed curves γ_1 and $\gamma_2 : S^1 \times [0, t_0) \to \mathbb{R}^2$ in the metric-affine plane be solutions of (3) and $\gamma_2(\cdot, 0)$ lie in the domain enclosed by $\gamma_1(\cdot, 0)$. Then $\gamma_2(\cdot, t)$ lies in the domain enclosed by $\gamma_1(\cdot, t)$ for all $t \in [0, t_0)$.

Proof. Set $\widetilde{S} = S_1 - S_2$ for two solutions and apply the **maximum principle** to a parabolic equation $\widetilde{S}_t = k_1 k_2(\widetilde{S}_{\theta\theta} + \widetilde{S})$.



Flowing inside curve vanishes before the outside curve vanishes.

Lemma 5

Let γ_t be a solution of (3) in the metric-affine plane with Ψ given in (5). Then $\tilde\gamma_t = \gamma_t + t[a_{21}, a_{12}]$ is a solution of (3) with the $\bar\nabla$-curvature $\bar k = k + \tilde\Psi$ and $\tilde\Psi = a_{30}\sin^3\theta + a_{03}\cos^3\theta$.

From Lemma 5 we conclude the following.

Proposition 5

If $a_{30} = a_{03} = 0$, see (7), then the problem (3) in the metric-affine plane reduces to the classical problem (1) in the Euclidean plane for modified by parallel translation of γ_t curves $\tilde\gamma_t = \gamma_t + t[a_{21}, a_{12}]$.

Solution with $a_{30} = a_{03} = 0$

One may show that

$$S(\theta, t) = \rho(t) - \epsilon_1(t)\sin\theta - \epsilon_2(t)\cos\theta$$

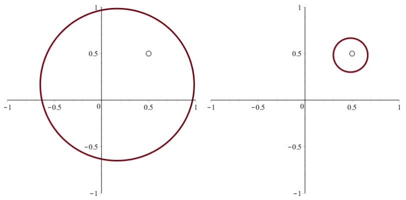
with $0 \le t \le \frac{1}{2}\,\rho^2(0)$ and

$$\rho(t) = \sqrt{\rho^2(0) - 2t}, \quad \epsilon_1(t) = a_{12}\,t, \quad \epsilon_2(t) = a_{21}\,t, \qquad (13)$$

is the support function of a solution of (3) with $a_{30} = a_{03} = 0$.
By (9), $S(\theta, t)$ corresponds to a family of circles

$$\gamma_t = [\,\rho(t)\cos\theta - \epsilon_2(t),\ \rho(t)\sin\theta - \epsilon_1(t)\,] \to \{point\}$$

with centers $C_t = -(\epsilon_2(t), \epsilon_1(t))$
and curvature $k = \frac{1}{\rho(t)}$.



(a) **Projective connections** $\bar\nabla = \nabla + \mathfrak{T}$ are defined by

$$\mathfrak{T}_X Y = \langle U, Y\rangle X + \langle U, X\rangle Y,$$

where U is a given vector field, Then $\Psi = \langle \mathfrak{T}_T T, N\rangle = 0$, see (5). Thus, (3) in the metric-affine plane with a projective connection is equal to (1) in the Euclidean plane.

(b) **Semi-symmetric connections** $\bar\nabla = \nabla + \mathfrak{T}$ are defined by

$$\mathfrak{T}_X Y = \langle U, Y\rangle\, X - \langle X, Y\rangle\, U,$$

where U is a given vector field (K. Yano). Such connections are **metric compatible**, and for them Frenet–Serret formulas (8) are valid. The definition (5) reads

$$\Psi = -\langle U, N\rangle = -\langle U, e_1\rangle \cos\theta - \langle U, e_2\rangle \sin\theta.$$

Then $a_{30} = \langle U, e_2 - e_1\rangle = -a_{03}$. Let U be a constant vector field on $\mathbb{R}^2$, then take an orthonormal frame $\{e_1, e_2\}$ in $(\mathbb{R}^2, g, \bar\nabla)$ such that $U \perp e_1 - e_2$. The problem (3) with a semi-symmetric $\bar\nabla$ and constant U reduces to the problem (1) in the Euclidean plane.

Proposition 6

Let a convex closed curve γ_0 in the metric-affine plane with condition $k_0 > 2\,c$, see (12), be evolved by (3). Then, the solution γ_t must be singular at some time $\omega > 0$.

Proof. We can put $a_{21}{=}a_{12}{=}0$. Hence, $\Psi = a_{30}\sin^3\theta + a_{03}\cos^3\theta$. Using the rotation $\theta \to \theta - \theta_0$ and Lemma 5, this reduces to

$$\Psi = \tilde a \sin^3\theta \quad (\text{for some } \tilde a < 0).$$

Let γ_0 lie in the circle Γ_0 of radius $\rho(0) \ge \max_{\theta\in S^1} \frac{S(\cdot,0)}{1-2c/k_0}$ and centered at the origin. Evolving Γ_0 by (3), we get a solution $\Gamma(\cdot, t)$ with support function S^Γ. By Proposition 4, γ_t lies in the domain enclosed in $\Gamma(\cdot,t)$, thus $S \le S^\Gamma$. Consider 2 families of circles

$$\Gamma^\pm_t = [\rho(t)\cos\theta,\ \rho(t)\sin\theta \pm t\tilde a], \quad \rho(t) = \sqrt{\rho^2(0) - 2\,t},$$

being solutions of (3), hence, their support functions satisfy (11),

$$\partial S^\pm/\partial t = ((S^\pm)''_{\theta\theta} + S^\pm)^{-1} \mp t\tilde a \sin\theta = \rho(t) \mp t\tilde a\sin\theta.$$

Finite time existence

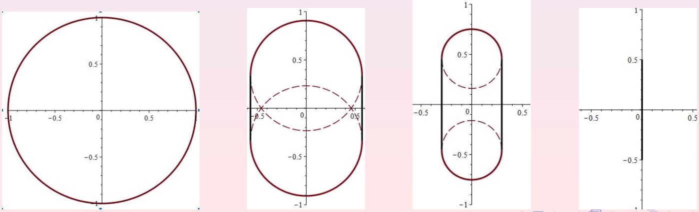
By Proposition 4, $\partial S/\partial t \le \partial S^\Gamma/\partial t$. Since $|\sin^3\theta| \le |\sin\theta|$ we get

$$\partial S^\Gamma/\partial t \le \begin{cases} \partial S^+/\partial t, & 0 \le \theta \le \pi, \\ \partial S^-/\partial t & \pi \le \theta \le 2\pi. \end{cases}$$

Thus, Γ_t lies (in $\mathbb{R}^2$) below tangent lines to upper semicircle of Γ_t^+ and above tangent lines to lower semicircle of Γ_t^-; thus,

$$\Gamma_t \subset \mathrm{conv}(\Gamma_t^+ \cup \Gamma_t^-).$$

The solution $\Gamma^\pm(\cdot, t)$ exists at a finite t-interval $[0,\tau]$, $\tau = \frac{1}{2}\rho^2(0)$, and converges, as $t \to \tau$, to a point $\Gamma_\tau^\pm = [0, \pm\tilde{a}\tau]$. Hence, the convex hull of $\Gamma_t^+ \cup \Gamma_t^-$ shrinks to the line segment with endpoints $(0, \pm\tilde{a}\tau)$. Thus, the solution γ_t must be singular at some $\omega \le \tau$. $\square$



Lemma 6

Let a convex closed curve γ_0 in the metric-affine plane with condition $k_0 > 2\,c$, see (12)*, be evolved by* (3)*. Then the area enclosed by $\gamma(\cdot,\omega)$ must be zero, i.e., $\gamma(\cdot,\omega)$ is either a point or a line segment.*

To complete the proof of Theorem 2, we note that if the flow (3) does not converge to a point as the enclosed by $\gamma(\cdot, t)$ area tends to zero (see Lemma 6), then $\min\limits_{\theta\in S^1} k(\theta, t)$ tends to zero as $t \uparrow \omega$.
But, by Proposition 3, the curvature of $\gamma(\cdot, t)$ has a uniform positive lower bound. So the flow converges to a point.

The area enclosed by the convex curve $\gamma(\cdot, t) \subset \mathbb{R}^2$ is calculated by

$$A(t) = -\frac{1}{2} \int_{\gamma(\cdot,t)} \langle \gamma(\cdot, t), N \rangle \, ds = \frac{1}{2} \int_0^{2\pi} \frac{S}{k} \, d\theta. \qquad (14)$$

We will estimate the maximal time ω under rather stronger condition to control convexity.

Proposition 7

Let a convex closed curve γ_0 in the metric-affine plane be evolved by (3). If $k_0 > 3c$ then the maximal time ω is estimated by

$$\omega \leq \frac{A(0)}{2\pi} \cdot \frac{k_0 - 2c}{k_0 - 3c}. \qquad (15)$$

Proof. Using the identity $\int_0^{2\pi} S(\bar{k}''_{\theta\theta}+\bar{k})\,d\theta=\int_0^{2\pi}(S''_{\theta\theta}+S)\,\bar{k}\,d\theta$, we get

$$\frac{d}{dt}A(t)=\frac{1}{2}\int_0^{2\pi}\frac{S'_t k - S k'_t}{k^2}\,d\theta=-\frac{1}{2}\int_0^{2\pi}\big[1+\Psi/k+S(\bar{k}''_{\theta\theta}+\bar{k})\big]\,d\theta$$
$$=-\frac{1}{2}\int_0^{2\pi}\big[1+\Psi/k+(S''_{\theta\theta}+S)\bar{k}\,\big]\,d\theta=-2\pi-\int_0^{2\pi}\frac{\Psi(\theta)}{k(\theta,t)}\,d\theta.$$

Using $k(\theta,t)\ge k_0-2c$ (Lemma 3) and $|\,\Psi|\le c$, see (6), we get

$$\frac{d}{dt}A(t)\le -2\pi+\frac{2\,\pi c}{k_0-2c}.$$

By this, $A(0)\ge 2\pi\,\omega\,\frac{k_0-3c}{k_0-2c}$. Hence, (15) is valid when $k_0>3c$. $\square$

Question: can one estimate ω when $2\,c<k_0\le 3\,c$?

One may study these questions and several related problems on flows in **metric-affine geometry**:

- The flow (3) for non-constant contorsion tensor $\mathfrak{T}$ of "small" C^1-norm and for not just convex γ_0 (or for $\bar{k} > 0$).
- Behavior of solutions for (3) and numerical experiments when γ_0 is immersed (as in [4]).
- The mean curvature flow in $\mathbb{R}^n$ ($n > 2$) with $\bar{\nabla}$.

THANK  YOU !