Estimating high order derivatives of a function through geometry and topology of its zero set

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1 Introduction

Let $f(x)$ be a smooth function on the unit n-dimensional ball $B^n \subset \mathbb{R}^n$.

A "rigidity inequality" for f is an explicit lower bound for the $(d+1)$-st derivative of f, which holds, if f exhibits certain patterns, forbidden for polynomials of degree d.

Smooth rigidity inequalities appear in Singularity Theory, Approximation Theory, Real Algebraic Geometry, Whitney extension problems,...

Quantitative Singularity Theory, example: Spike-train super-resolution in Signal Processing.

Consider a "spike-train" signal

$$F(x) = \sum_{j=1}^{d} a_j \delta\left(x - x_j\right) + N(x) \qquad (1.1)$$

We know the form (1.1) of the signal, and a few moment measurements

$$\mu_k = \int F(x) x^k dx.$$

But the specific parameters a_j and x_j (and usually also d!) are the unknowns to be found.

This leads to the classical *Prony system, 1795*:

$$\sum_{j=1}^{d} a_j x_j^k = \mu_k, \ k = 0, 1, \ldots, 2d-1. \qquad (1.2)$$

Main difficulty: near-collision of the nodes x_i (*Spike-train super-resolution problem*). Complicated algebraic singularities + strong noise $N(x)$. Our group works by now for 15 years.

Hope: Quantitative Singularity Theory can help.

Example. A nonzero univariate polynomial $P(x)$ of degree d can have at most d real zeros.

Proposition 1.1 *For each* $(d+1)$*-smooth function* $f(x)$ *on* $[-1,1]$*, with* $\max_{[-1,1]}|f(x)| \geq 1$ *and with* $d+1$ *or more distinct zeroes on* $[-1,1]$*, we have*

$$\max_{[-1,1]}|f^{(d+1)}(x)| \geq \frac{(d+1)!}{2^{d+1}}.$$

A short proof can be given using the Lagrange remainder formula in the polynomial interpolation of f on its zeroes , or (more or less equivalently) via divided finite differences.

Our goal: higher-dimensional versions of this result. There are many ways to interpret conditions of Proposition 1.1 in higher dimensions, in particular, as *density conditions*, or, alternatively, as *topological conditions*. Results in both directions follow.

Sometimes, one can use line sections:

Theorem 1.1 *(Y. 84) Let $f(x)$ be a smooth function on B^n, with* $\max_{B^n}|f| = 1$.

If for some $d \geq 1$, $\max_{B^n}||f^{(d)}|| \leq 2^{-d-1}$, *then*

1. The set of zeroes Z of f is contained in a countable union of smooth hypersurfaces.

2. The $(n-1)$-Hausdorff measure of Z is bounded by a constant, depending only on n and d.

Proof: Restrict f to the straight lines, apply Proposition 1.1, and use Integral Geometry.

Problem 1: This approach works only when "many" straight lines intersect Z at "many" points.

Via integral-geometric interpretation, this is the case of sets $Z \subset B^n$, containing hypersurfaces of a big Hausdorff $(n-1)$-measure (like in Theorem 1.1).

A "backward reading" of this theorem provides the corresponding rigidity inequality.

Still, for many natural classes of sets Y, there are only a few intersection points with generic straight lines.

For example, this is the case for Z being a finite set (unless, by a rare coincidence, many points of Z lie on the same straight line).

Need alternative approaches!

Problem 2: Multidimensional divided finite differences (on arbitrary sets) provide only a very limited information on the high-order derivatives.

This is one of the main problems also in Whitney extension, as well as in Interpolation Theory and Signal processing.

Recently some new "multi-dimensional" approaches to smooth rigidity appeared:

1. [Ler.Ste] *What is the degree of a smooth hypersurface?* Here it was shown that a transversal singular set of a smooth function is isotopic to a corresponding singular set of a certain polynomial, of an explicitly bounded degree.

2. [Y1] *Smooth rigidity and Remez-Type Inequalities*, Anal.Math.Phys. 11, 89 (2021). The d-rigidity of a set Z is approximately the reciprocal d-Remez constant of Z.

3. [Y2] *Smooth rigidity and Remez inequalities via Topology of level sets* (Similar to [Ler.Ste], but no transversality assumptions, in a very special situation).

3. [Y3] *Smooth rigidity via probe curves.* We replace straight lines with smooth curves ω. This requires analysis of the high order chain-rule expressions, on one side, and construction of curves ω with small high-order derivatives, passing through some $d+1$ distinct points in Z, on the other side.

2 Smooth rigidity - definition, background

Let $f : B^n \to \mathbb{R}$ be a $d+1$ times continuously differentiable function on the unit closed ball $B^n \subset \mathbb{R}^n$.

For $l = 0, 1, \ldots, d+1$ put

$$M_l(f) = \max_{z \in B^n} \|f^{(l)}(z)\|,$$

where the norm of the l-th derivative $f^{(l)}(z)$ of f is defined as the sum of the absolute values of all the partial derivatives of f of order l.

For $Z \subset B^n$ let $W_d(Z)$ denote the set of C^{d+1} smooth functions $f(z)$ on B^n, vanishing on Z, with $M_0(f) = 1$.

Definition 2.1 *For $Z \subset B^n$ we define the d-th rigidity constant $\mathcal{RG}_d(Z)$ as*

$$\mathcal{RG}_d(Z) = \inf_{f \in W_d(Z)} M_{d+1}(f).$$

In fact, this is a very special case of one of the central notions in the Whitney smooth extension theory.

In terms of the rigidity constant $\mathcal{RG}_d(Z)$ Proposition 1.1 implies

Proposition 2.1 *For any $Z \subset B^1$ we have $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$, if Z consists of at least $d+1$ different points, and $\mathcal{RG}_d(Z) = 0$ if Z consists of at most d different points.*

Thus, in dimension one, $\mathcal{RG}_d(Z)$ is either zero or at least $\frac{(d+1)!}{2^{d+1}}$.

In higher dimensions, Proposition 1.1 implies, via line sections:

Proposition 2.2 *For any $Z \subset B^n$ with a non-empty interior, and for any d,*

$$\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}.$$

This reminds one-dimensional situation: $\mathcal{RG}_d(Z)$ cannot be arbitrarily small for sets Z with a non-empty interior. However, in higher dimensions one can easily produce finite Z with arbitrary small nonzero $\mathcal{RG}_d(Z)$.

Problem Are there other classes of sets Z, besides those with a non-empty interior, for which Proposition 2.2 still holds? Discrete (finite) sets Z?

Let $M(\epsilon, Z)$ denote the minimal number of ϵ-balls, covering Z. The Minkovski (box, entropy) dimension of Z is defined as

$$\dim_e Z = \limsup_{\epsilon \to 0} \frac{\log M(\epsilon, Z)}{\log \epsilon}.$$

Theorem 2.1 *([Y3]) Let $Z \subset B^n$. If $\dim_e Z > n - \frac{1}{d+1}$ then*

$$\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{2d+1}}.$$

This theorem should be still written with t, not T.

Thus the "rigidity jump" is the effect of the density of Z. It works for finite sets also.

Sketch of the proof If for a given $f \in W_d(Z)$ we could find a straight line ℓ, passing through a point z_0 where $|f(z_0)| = 1$, and through certain $d+1$ points in Z (where f vanishes), the result would follow from Proposition 1.1.

We replace straight lines with smooth curves ω. This requires analysis of the high order chain-rule expressions, elementary, but pretty cumbersome.

Next we construct curves ω with small high-order derivatives, passing through z_0 and through some $d+1$ distinct points in Z.

This is done in two steps:

First, using "discrete integral geometry" we find straight lines ℓ through z_0 passing "near" some $d+1$ distinct points $z_1, \ldots, z_{d+1}$ in Z. Here, of course, a high density of Z is used.

Then we deform ℓ into a curve ω, forcing it to pass exactly through $z_1, \ldots, z_{d+1}$, while preserving small size of high-order derivatives of ω.

3 Rigidity and Remez constant

Definition 3.1 *For a set $Z \subset B^n \subset \mathbb{R}^n$ and for each $d \in \mathbb{N}$ the Remez (norming, Lebesgue ...) constant $\mathcal{R}_d(Z)$ is the minimal K for which the inequality*

$$\sup_{B^n} |P| \leq K \sup_{Z} |P|$$

is valid for any real polynomial $P(x) = P(x_1, \ldots, x_n)$ of degree d.

We always have $\mathcal{R}_d(Z) \geq 1$.

For some Z the Remez constant $\mathcal{R}_d(Z)$ may be equal to ∞. In fact, $\mathcal{R}_d(Z)$ is infinite if and only if Z is contained in the set of zeroes

$$Y_P = \{x \in \mathbb{R}^n, \mid P(x) = 0\}$$

of a certain polynomial P of degree d.

Denote $\hat{\mathcal{R}}_d(Z)$ the reciprocal Remez constant

$$\hat{\mathcal{R}}_d(Z) := \frac{1}{\mathcal{R}_d(Z)}.$$

Theorem 3.1 *([Y1]) For any $Z \subset B^n$, we have*

$$\frac{(d+1)!}{2}\hat{\mathcal{R}}_d(Z) \leq \mathcal{RG}_d(Z).$$

A lot is known about the Remez constant. Theorem 3.1 translates this information to the rigidity constant. Lower bounds for $\hat{\mathcal{R}}_d(Z)$ can be given in terms of the volume of Z, of its density, and its topology.

In many cases the inequality of Theorem 3.1 is not sharp. In particular, for Z with a non-empty interior, $\hat{\mathcal{R}}_d(Z)$ can be arbitrarily small, while $\mathcal{RG}_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$.

On the other hand, Theorem 3.1 provides a non-trivial bound for each Z with $\dim_e Z > n-1$, while theorem 2.1 requires $\dim_e Z > n - \frac{1}{d+1}$. Is there a "critical dimension"?

One conclusion for rigidity is:

Theorem 3.2 *(Y1) Let $Z \subset B^n$ be a finite set, and let ρ be the minimal distance between the points of Z. Assume that the cardinality $M = |Z|$ satisfies $M > (4d)^n(\frac{1}{\rho})^{n-1}$. Then*

$$0 < \frac{(d+1)!}{2}\left(\frac{M\rho^n - (4d)^n\rho}{4n}\right)^d \leq \mathcal{RG}_d(Z).$$

Informally, if in resolution ρ the set Z "looks more massive than an algebraic hypersurface of degree d", then its d-th rigidity constant is positive.

4 Rigidity and Remez via topology of Z

For a given d let $Z \subset B^n$ be a smooth compact hypersurface with $N = (d-1)^n + 1$ connected components Z_j.

Assume that the interiors U_j of Z_j are disjoint. Let μ_j be the n-volume of the interior of Z_j, and put

$$\mu = \min \mu_j, \ j = 1, \dots, N.$$

Theorem 4.1 *([Y2]) For each polynomial P of degree d on $\mathbb{R}^n$ we have*

$$\frac{\max_{B^n}|P|}{\max_Z|P|} \leq (\frac{4n}{\mu})^d,$$

and hence

$$\hat{\mathcal{R}}_d(Z) \geq (\frac{\mu}{4n})^d, \qquad \mathcal{RG}_d(Z) \geq \frac{1}{(d+1)!}(\frac{\mu}{4n})^d.$$

Sketch of the proof: the classical Brudny-Ganzburg-Remez inequality provides for any Z

$$\hat{\mathcal{R}}_d(Z) \geq (\frac{m(Z)}{4n})^d, \tag{4.1}$$

where $m(Z)$ is the n-volume of Z.

However, the n-volume of our hypersurface Z is zero, so the above bound does not help.

Assume, in contradiction with Theorem 4.1, that there is a polynomial P of degree d with $\max_{B^n}|P| = 1$, and $\max_Z|P| < (\frac{\mu}{4n})^d$. Applying (4.1) to P and to each interior set U_j, we get (since $\max_{B^n}|P| = 1$)

$$\max_{U_j}|P| \geq (\frac{\mu_j}{4n})^d \geq (\frac{\mu}{4n})^d, \quad j = 1, \ldots, N.$$

But on the boundary Z_j, by assumptions, $|P|$ is strictly smaller. We conclude that inside U_j the polynomial P has a local maximum or minimum, i.e, a critical point. Thus we get at least $N = (d-1)^n + 1$ critical points of P, which contradicts the Bezout theorem.

5 What is the degree of a smooth hypersurface (A. Lerario and M. Stecconi, [Ler.Ste])

Let $f : B^n \to \mathbb{R}^q$ be a smooth mapping. For W a closed semialgebraic subset of the jet space $J^r(B^n, \mathbb{R}^q)$ we consider the (type W) singularity $Y_W(f)$, defined as

$$Y_W(f) = (j^r f)^{-1}(W),$$

$j^r f : B^n \to J^r(B^n, \mathbb{R}^q)$ being the r-jet extension of f.

Define the discriminant set Δ_W as the set of all $f \in C^{r+1}$, whose r-jet extension $j^r f : B^n \to J^r(B^n, \mathbb{R}^q)$ is not transversal to W. Quantitatively, the transversality of f to W is measured by the distance $\delta_W(f)$ from f to Δ_W in the space of C^{r+1}-smooth functions.

Theorem 5.1 *([Ler.Ste], Theorem 1) Let $W \subset J^r(B^n, \mathbb{R}^q)$ be closed semialgebraic. For every $f \in C^{r+2}(B^n, \mathbb{R}^q)$ with $j^r f$ transversal to W there exists a polynomial map $P = (P_1, \ldots, P_q)$ with*

$$\deg(P_i) \leq C(r)\max\left\{r+1, \frac{||f||_{C^{r+2}}}{\delta_W(f)}\right\},$$

such that the (type W) singularity $Y_W(f)$ is smoothly isotopic to $Y_W(P)$.

This result allows the authors to immediately bound the topological complexity of $Y_W(f)$ ([Ler.Ste], Corollary 2).

From the smooth rigidity point of view, Theorem 5.1 immediately implies the following result, where we put $\bar{C}(r) = \frac{1}{C(r)}$:

Theorem 5.2 *For f as above, and for a certain closed semialgebraic $W \subset J^r(B^n, \mathbb{R}^q)$, assume that the (type W) singularity $Y_W(f)$ is not smoothly isotopic to $Y_W(P)$ for any polynomial map P of a given degree $d \geq r+1$. Then*

$$||f||_{C^{r+2}} \geq \bar{C}(r) \cdot d \cdot \delta_W(f).$$

Of course, the condition of $Y_W(f)$ not being smoothly isotopic to $Y_W(P)$ for any polynomial map P of degree $d \geq r+1$ can be replaced by stronger conditions in terms of topological complexity of $Y_W(P)$ (for example, in terms of the sum of its Betti numbers).

If we take the degree d as the "measure of complexity" of polynomials, then Theorem 5.2 implies the *linear grows of the norm* $||f||_{C^{r+2}}$ *with* d, at least. In would be interesting to make this very initial observation more rigorous.

We believe that an explicit and detailed study, in the above directions, of the *specific* singularity types W is well justified. As an example, consider the following statement:

Let f be a C^{r+2}-smooth function on B^n. Assume that for certain $d \geq r+2$ the function f has at least $(d-1)^n+1$ non-degenerate critical points, with the minimal eigenvalue of the Hessian $H(f)$ at each critical point being at least $\gamma > 0$ in absolute value.

Then $||f||_{C^{r+2}} \geq \bar{C}(r) \cdot d \cdot \gamma$.

This is a special case of Theorem 5.2. I do not see any other way to prove it, but via [Ler.Ste]. And I am sure this and similar very specific special cases of [Ler.Ste] (like Whitney plane singularities) deserve careful study.

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