SUBGROUPS OF $\mathrm{PSL}_2(\mathbb{C})$ WHICH ARE EXTREME FOR DISCRETENESS CONDITIONS

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It is well-known that $\mathrm{PSL}(2,\mathbb{C})$ is isomorphic to the full group of orientation-preserving isometries of the hyperbolic 3-space. Thus, the question about discreteness of a given subgroup is actual in algebraic and geometric contexts both.

In the present talk we will discuss two-generated groups which are extreme for some discreteness conditions. We will mostly interested in groups which uniformize hyperbolic 3-manifolds arising as knot complements and related 3-orbifolds.

For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ let us denote

$$\mathcal{J}(f,g) = |\mathrm{tr}^2(f) - 4| + |\mathrm{tr}[f,g] - 2|.$$

Let $G < \mathrm{PSL}(2,\mathbb{C})$ be a 2-generated non-elementary group. The value

$$\mathcal{J}(G) = \inf_{\langle f,g\rangle = G} \mathcal{J}(f,g),$$

is said to be a *Jørgensen number* of G. Jørgensen numbers originally arise in the following discreteness condition [1]: if non-elementary group G is discrete then $\mathcal{J}(G) \geq 1$. It was shown in [2] that the figure-eight knot complement is the unique hyperbolic 3-manifold whose fundamental group has Jørgensen number equals to one. Jørgensen numbers for some 2-bridge knot groups were calculated in [2].

Let us denote by K the figure-eight knot and by $K(n)$ the hyperbolic 3-orbifold with singular set K and singular angle $2\pi/n$, $n \geq 4$. The knot group has the presentation

$$\pi_1(S^3 \setminus K) = \langle\, f, g \mid [g,f]\, g^{-1} = f\, [g,f] \,\rangle$$

and the orbifold group has the following presentation:

$$\pi^{\mathrm{orb}}(K(n)) = \langle\, f_n, g_n \mid f_n^n = g_n^n = 1,\ [g_n, f_n]\, g_n^{-1} = f_n\, [g_n, f_n] \,\rangle.$$

Both of them have faithful representations in $\mathrm{PSL}(2,\mathbb{C})$.

We will describe behavior of Jørgensen numbers of figure-eight knot orbifold groups.

Theorem 1. [3] *Let $n \geq 4$. Then the following inequalities hold:*

$$1 \leq \mathcal{J}(\pi^{\mathrm{orb}}(K(n))) \leq 4\sin^2(\pi/n) + \sqrt{1 + 4\sin^2(\pi/n)}.$$

Corollary 1. [3] *The following convergence holds:*

$$\lim_{n\to\infty} \mathcal{J}(\pi^{\mathrm{orb}}(K(n))) = \mathcal{J}(\pi_1(S^3 \setminus K)).$$

An analog of a Jørgensen number was introduced in [4] and [5]. For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ such that $\mathrm{tr}[f,g] \neq 1$ denote $\mathcal{G}(f,g) = |\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f,g] - 1|$. Let $G < \mathrm{PSL}(2,\mathbb{C})$ be a 2-generated group. The value

$$\mathcal{G}(G) = \inf_{\langle f,g\rangle = G} \mathcal{G}(f,g),$$

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is said to be a *GMT number* of G. Gehring and Martin [4] and independently Tan [5] proved that if G is discrete then $\mathcal{G}(G) \geq 1$.

The following results demonstrate behavior of GMT numbers of figure-eight knot orbifold groups.

Theorem 2. [3] *The following equality holds:* $\mathcal{G}(\pi_1(S^3 \setminus K)) = 3$.

Theorem 3. [3] *Let* $n \geq 4$. *Then the following inequalities hold:*

$$1 \leq \mathcal{G}(\pi^{\mathrm{orb}}(K(n))) \leq 3 - 4\sin^2(\pi/n).$$

Corollary 2. [3] *The following equality holds:* $\mathcal{G}(\pi^{\mathrm{orb}}(K(4))) = 1$.

References

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