ON DISCRETENESS OF 2-GENERATOR SUBGROUPS OF $\mathrm{PSL}(2,\mathbb{C})$

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Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, denote

$$\mathrm{tr}(M) = a + d \quad \text{and} \quad \|M\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

A matrix $M \in \mathrm{SL}(2,\mathbb{C}) \setminus \{\pm \mathrm{Id}\}$ is called elliptic if $\mathrm{tr}^2(M) \in [0;4)$, parabolic if $\mathrm{tr}^2(M) = 4$, loxodromic if $\mathrm{tr}^2(M) \in \mathbb{C} \setminus [0;4]$.

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Recall that $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm \mathrm{Id}\}$.

Definition

An element $g \in \mathrm{PSL}(2,\mathbb{C})$ is called *elliptic*, *parabolic*, or *loxodromic* if so is its representative in $\mathrm{SL}(2,\mathbb{C})$.

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Endow $\mathrm{PSL}(2,\mathbb{C})$ with the quotient topology of the norm $\|\cdot\|$.

Definition

A group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be *discrete* if G is a discrete set.

Let $\mathbb{H}^3$ be the Poincaré half-space model of hyperbolic 3-space, i.e. the set $\left\{(z, t) \,\middle|\, z = x + yi \in \mathbb{C}, t > 0\right\}$ with the metric $ds^2 = \dfrac{|dz|^2 + dt^2}{t^2}$.

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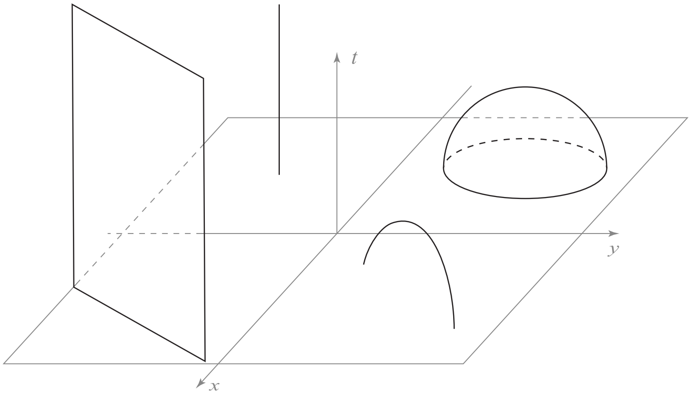

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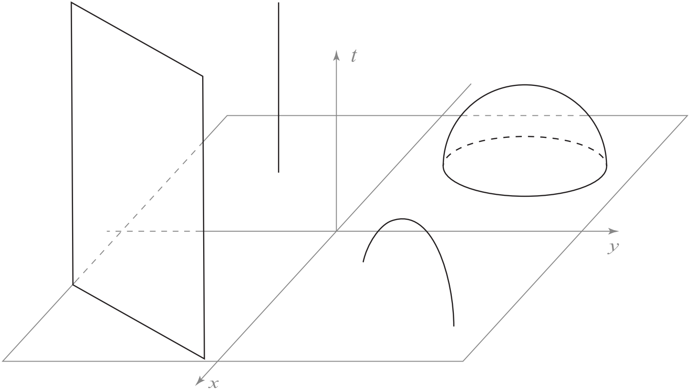
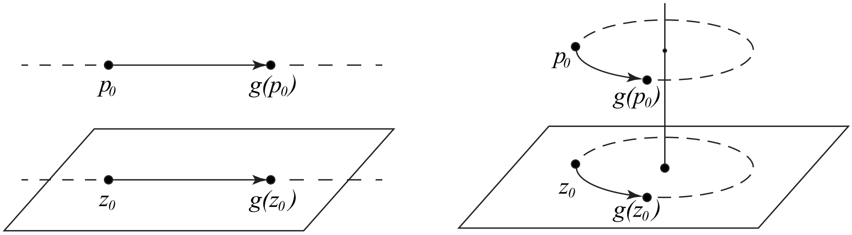


Identify $\partial\mathbb{H}^3$ with $\overline{\mathbb{C}}$. It is known that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathsf{PSL}(2,\mathbb{C})$.

A parabolic element g is a rotation about the point at infinity.

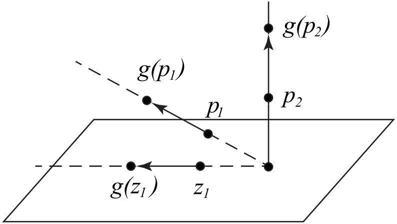


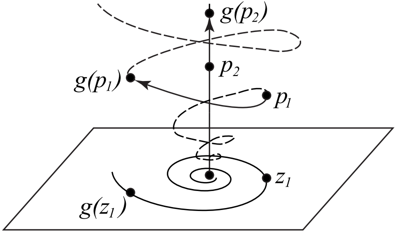
Definition

Let g be a nonparabolic element. The geodesic in $\mathbb{H}^3$ joining two fixed points of g in $\overline{\mathbb{C}}$ is called the *axis* of g and is denoted by ℓ_g.

An elliptic element g is a rotation about ℓ_g by an angle $\varphi_g \in (-\pi; \pi]$.

A loxodromic element g is the composition of a translation along ℓ_g by an amount $\tau_g \geq 0$ and a rotation about ℓ_g by an angle $\varphi_g \in (-\pi;\pi]$.





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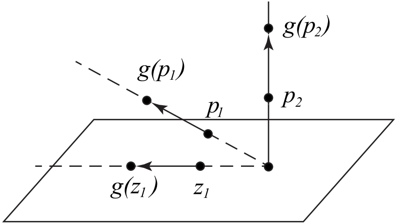

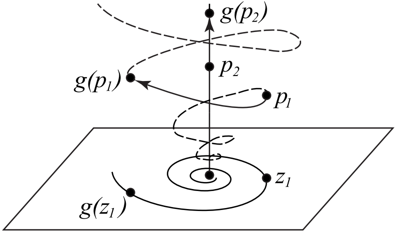

Definition

A group $G < \mathrm{PSL}(2,\mathbb{C})$ is called *elementary* if there exists a finite G-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. Otherwise, a group G is called *nonelementary*.

Theorem 1 (Jørgensen, 1977)

A nonelementary group $G < \mathrm{PSL}(2, \mathbb{C})$ is discrete if and only if each 2-generator subgroup of G is discrete.

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Let $f, g \in \mathrm{PSL}(2, \mathbb{C})$ and $\langle f, g \rangle$ be the group generated by the elements f and g.

Definition

By the *parameters* of $\langle f, g \rangle$ we mean three complex numbers
$\gamma = \gamma(f, g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2,\ \ \beta = \beta(f) = \mathrm{tr}^2(f) - 4,\ \ \beta' = \beta(g) = \mathrm{tr}^2(g) - 4.$

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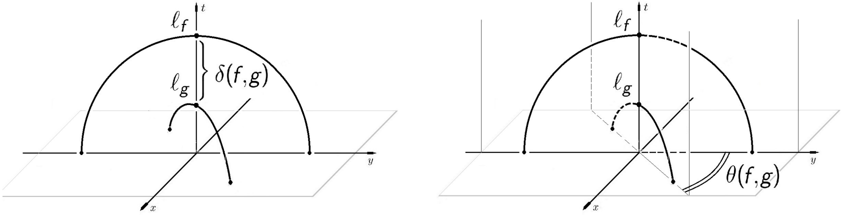
Proposition 1 (Gehring — Martin, 1992)

Let $h \in \mathrm{PSL}(2,\mathbb{C})$ be a nonparabolic element and τ_h be a translation length of h.
Then $\mathrm{ch}(\tau_h) = \dfrac{|\beta(h)+4| + |\beta(h)|}{4}$ and $\cos(\varphi_h) = \dfrac{|\beta(h)+4| - |\beta(h)|}{4}$.

Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$ be nonparabolic elements which have no common fixed points in $\overline{\mathbb{C}}$. Denote the angle and the distance between their axes by $\delta(f,g)$ and $\theta(f,g)$, respectively.



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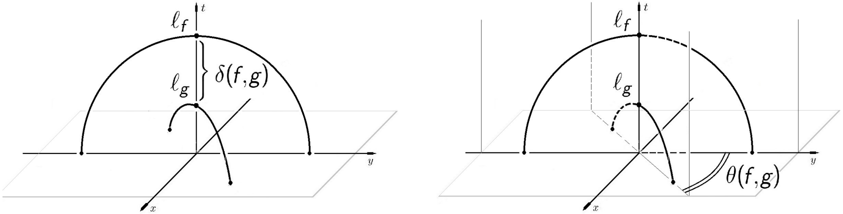

Proposition 2 (Gehring — Martin, 1994)

Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$ be elements as above. Then

$$\mathrm{ch}\big(2\delta(f,g)\big) = \left|\frac{4\gamma}{\beta\beta'} + 1\right| + \left|\frac{4\gamma}{\beta\beta'}\right|, \quad \cos\big(2\theta(f,g)\big) = \left|\frac{4\gamma}{\beta\beta'} + 1\right| - \left|\frac{4\gamma}{\beta\beta}\right|.$$

Theorem 2 (Jørgensen, 1976, 1979)

Let $\langle f, g \rangle < \mathrm{PSL}(2,\mathbb{C})$ be a nonelementary discrete group. Then

$$|\gamma| + |\beta| \geq 1 \quad \text{and} \quad |\gamma - \beta| + |\beta| \geq 1.$$

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Theorem 3 (Gehring — Martin, 1994)

Let $\langle f, g\rangle < \mathrm{PSL}(2,\mathbb{C})$ be a nonelementary discrete group such that f and g are elliptic elements of orders $m \geq 3$ and $n \geq 2$, respectively. Then

$$\delta(f,g) \geq \frac{1}{2} \cdot \mathrm{Arch}\left(\frac{1+\sqrt{5}}{3}\right) = 0,197\ldots \, .$$

Theorem 4 (Rasskazov, 2006)

Let $G = \langle f, g \rangle < \mathrm{PSL}(2, \mathbb{C})$ a group be such that **f and g are elliptic elements of orders $m \geq 3$ and $n \geq 2$**, respectively. Suppose that the following inequality holds:

$$\operatorname{ch} \delta(f, g) \geq \frac{\cos(\pi/m)\cos(\pi/n)\cos\theta(f, g) + 1}{\sin(\pi/m)\sin(\pi/n)}. \tag{1}$$

Then G is a nonelementary discrete group and $G = \langle f \rangle * \langle g \rangle$.

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Theorem 5 (M, 2014)

Let $G = \langle f, g \rangle < \mathrm{PSL}(2, \mathbb{C})$ be a group such that **f is an elliptic element of order $m \geq 3$** and **g is an involution,** respectively. Suppose that one of the following conditions holds:

$$\begin{array}{llll} (1) & 0 \leq \theta(f, g) \leq \pi/4 & \text{and} & \mathrm{sh}\,\delta(f, g) \geq \mathrm{ctg}(\pi/m)\cos\theta(f, g); \\ (2) & \pi/4 \leq \theta(f, g) \leq \pi/2 & \text{and} & \mathrm{sh}\,\delta(f, g) \geq \mathrm{ctg}(\pi/m)\sin\theta(f, g). \end{array}$$

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Theorem 6 (M, 2013)

Let $G = \langle f, g \rangle < \mathrm{PSL}(2, \mathbb{C})$ be a group such that **f and g are loxodromic elements** with translation lengths τ_f and τ_g. Write $\alpha_f = \arcsin\big(1/\operatorname{ch}(\tau_f/2)\big)$ и $\alpha_g = \arcsin\big(1/\operatorname{ch}(\tau_g/2)\big)$. Suppose that one of the following conditions holds:

$$(1) \quad \alpha_f + \alpha_g \leq \theta(f, g),$$

$$(2) \quad \alpha_f + \alpha_g > \theta(f, g) \quad \text{and} \quad \operatorname{ch}\delta(f, g) \geq \frac{\operatorname{ch}(\tau_f/2)\operatorname{ch}(\tau_g/2)\cos\theta(f, g) + 1}{\operatorname{sh}(\tau_f/2)\operatorname{sh}(\tau_g/2)},$$

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Let $G = \langle f, g \rangle < \mathrm{PSL}(2,\mathbb{C})$ be a group such that f **is an loxodromic element** with translation length τ_f and g **is an elliptic element of order** $n \geq 2$. Suppose that the following inequality holds:

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(1) $0 \leq \theta(f,g) \leq \pi/4$ and $\alpha_f \leq \theta(f,g)$,

(2) $0 \leq \theta(f,g) \leq \pi/4$, $\alpha_f > \theta(f,g)$ and $\operatorname{ch}\delta(f,g) \geq \operatorname{cth}(\tau_f/2)\cos\theta(f,g)$,

(3) $\pi/4 \leq \theta(f,g) \leq \pi/2$ and $\alpha_f \leq \pi/2 - \theta(f,g)$,

(4) $\pi/4 \leq \theta(f,g) \leq \pi/2$, $\alpha_f > \pi/2 - \theta(f,g)$ and $\operatorname{ch}\delta(f,g) \geq \operatorname{cth}(\tau_f/2)\sin\theta(f,g)$,

Then G is a nonelementary discrete group and $G = \langle f\rangle * \langle g\rangle$.

Maskit's question

Let $\langle f, g\rangle < \mathrm{PSL}(2,\mathbb{C})$ be a group such that an element f has two fixed points $z_1, z_2 \in \overline{\mathbb{C}}$ and $g(z_1) = z_2$. When is $\langle f, g\rangle$ a discrete group?

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Theorem 9 (Maskit, 1989)

If $\langle f, g\rangle < \mathrm{PSL}(2,\mathbb{C})$ is a discrete Maskit's group, then either g is an involution, or f is an elliptic element of order 2, 3, 4, or 6.

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Proposition 3 (Gehring — Martin, 1994)

The following two statements are equivalent:

(1) $\langle f, g \rangle < \mathrm{PSL}(2, \mathbb{C})$ is a Maskit's group,

(2) $\mathrm{par}\langle f, g \rangle = (\beta, \beta, \beta')$ and $\beta \neq 0$.

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Remark. Let $\langle f, g\rangle < \mathrm{PSL}(2,\mathbb{C})$ be a discrete group, $\mathrm{par}\langle f, g\rangle = (\beta, \beta, \beta')$ and $\beta \neq 0$. Then either $\mathrm{par}\langle f, g\rangle = (\beta, \beta, -4)$ and $\beta \neq 0$, or $\mathrm{par}\langle f, g\rangle = (\beta, \beta, \beta')$ and $\beta \in \{-4, -3, -2, -1\}$.

Theorem 10 (Maskit, Klimenko, 1989)

Let $\langle f, g\rangle < \mathrm{PSL}(2,\mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (-4, -4, \beta')$ and $\beta' \geq -4$. Then $\langle f, g\rangle$ is a discrete group if and only if $\beta' \in \{ -4\sin^2(\pi/n) \mid n = 2, 3, 4, \ldots\}$ or $\beta' \geq 0$.

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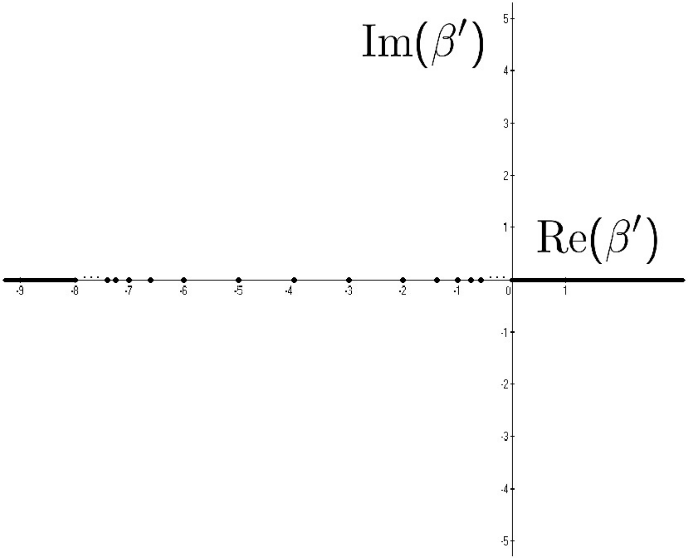
Corollary 5 (Gehring — Gilman — Martin, 2001)

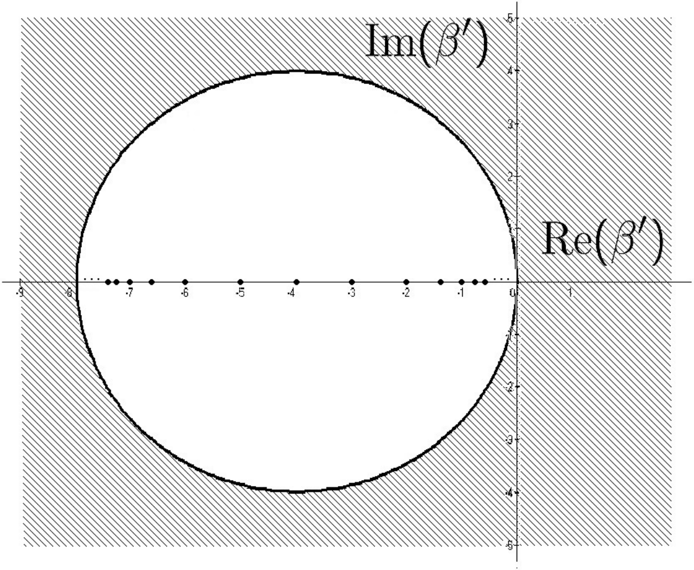
Let $\langle f, g\rangle < \mathrm{PSL}(2, \mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (-4, -4, \beta')$ and $\beta' < -4$. Then $\langle f, g\rangle$ is a discrete group if and only if $\beta' \in \{ -4\bigl(1 + \cos^2(\pi/n)\bigr) \mid n = 3, 4, \ldots\}$ or $\beta' \leq -8$.


Im(β')
Re(β')
5
4
3
2
1
-1
-2
-3
-4
-5
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1

Theorem 11 (M, 2014)

Let $\langle f, g\rangle < \mathrm{PSL}(2, \mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (-4, -4, \beta')$, $\beta' \in \mathbb{C} \setminus \mathbb{R}$ and $|\beta' + 4| \geq 4$. Then $\langle f, g\rangle$ is a discrete group.

Maskit's groups with parameters $(-4, -4, \beta')$



THANK YOU!