Subgroups of $\mathrm{PSL}_2(\mathbb{C})$ which are extreme for discreteness conditions

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Hyperbolic 3-manifolds and hyperbolic knots.

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$$g(z, t) = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2 t^2}, \frac{t}{|cz + d|^2 + |c|^2 t^2} \right).$$

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A knot $K \subset S^3$ is said to be hyperbolic if $S^3 \setminus K$ is a hyperbolic manifold.

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Matrix $A \in \mathrm{SL}(2,\mathbb{C}) \setminus \{\pm \mathrm{I}\}$ is said to be

- *elliptic* if $\mathrm{tr}^2(A) \in [0;4)$;
- *parabolic* if $\mathrm{tr}^2(A) = 4$;
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An element of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm \mathrm{I}\}$ is said to be *elliiptic*, *parabolic*, or *loxodromic* if its preimage in $\mathrm{SL}(2,\mathbb{C})$ is of such type.

Jørgensen numbers.

[Jørgensen, 1976] Let $\langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ be non-elementary and discrete. Then $|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}[f,g] - 2| \geq 1$. This low bound is sharp.

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Two-generated non-elementary discrete group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be *extreme* (also, *Jørgensen group*) if it can be generated by f and g such that $\mathcal{J}(f,g)=1$.

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Obviously, $\mathrm{PSL}(2,\mathbb{Z})$ is non-elementary and discrete. It is known that $\mathrm{PSL}(2,\mathbb{Z}) = \langle f, g \mid g^2 = (gf)^3 = 1 \rangle$, where

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $\mathrm{tr}^2(f) = 4$ and $\mathrm{tr}^2(g) = 0$, f is parabolic and g is elliptic of order two.

It is easy to check that $[f,g] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $\mathcal{J}(f,g) = |4-4| + |3-2| = 1$.

Elliptic and parabolic type extreme groups.

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Let $\langle f, g \rangle < \mathrm{PSL}(2, \mathbb{C})$ be extreme group and $\mathcal{J}(f, g) = 1$. Then $3 < \mathrm{tr}^2(f) \leq 4$.
Thus, either f is *elliptic* of order $n \geq 7$ or f is *parabolic*.

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[Jørgensen – Kiikka, 1975] The only extreme subgroups of $\mathrm{PSL}(2, \mathbb{R})$ are $(2, 3, n)$-*triangle groups* with $n \geq 7$ or $n = \infty$.

Observation.

Suppose $f, g \in \mathrm{PSL}(2,\mathbb{C})$ and f is parabolic. Up to a conjugation in $\mathrm{PSL}(2,\mathbb{C})$ we can assume that

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Then

$$\mathrm{tr}[f,g] = \mathrm{tr} \begin{pmatrix} 1 + ac + c^2 & * \\ * & 1 - ac \end{pmatrix} = 2 + c^2.$$

Therefore,

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Hence $\mathcal{J}(f,g) = 1$ if and only if $|c| = 1$.

Parabolic type extreme groups.

Suppose $\langle f, g \rangle$ is non-elementary and f be parabolic. Up to a conjugation in $\mathrm{PSL}(2,\mathbb{C})$ we can assume that

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g = g_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. Denote $G_{\sigma,\mu} = \langle f, g_{\sigma,\mu} \rangle$.

By the above observation $\mathcal{J}(f, g_{\sigma,\mu}) = 1$ if and only if $|\sigma| = 1$.

Suppose that $\sigma = -ie^{i\theta}$, $\theta \in [0, 2\pi)$.

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Conjecture. [Li – Oichi – Sato, 2005]
Every parabolic type extreme group is conjugated to $G_{-ie^{i\theta},ik}$, where $k \in \mathbb{R}$.

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All discrete groups of type $G_{-ie^{i\theta},ik}$, where $k \in \mathbb{R}$, are classified.

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[Callahan, 2009]
There are extreme groups which are not of the above type: $\mathrm{PGL}(2,O_3)$, $\mathrm{PSL}(2,O_3)$, $\mathrm{PSL}(2,O_7)$ and $\mathrm{PSL}(2,O_{11})$.
All non-compact arithmetic groups extreme groups are classified (there are 14).

Here O_d is a ring of integers of $\mathbb{Q}[\sqrt{-d}]$. Recall that $O_d = \mathbb{Z}[\omega]$,
where $\omega = \sqrt{-d}$ if $d \equiv 0,1,2 \mod 4$ and $\omega = (1+\sqrt{-d})/2$ if $d \equiv 3 \mod 4$.

Jørgensen subgroups of the Picard group.

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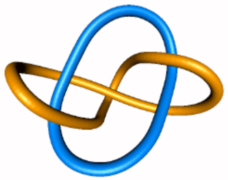
[F. Conzalez – Acuna, A. Romirez, 2007]
All subgroups G of $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-1}])$ with $\mathcal{J}(G) = 1$ are listed.

Consider the Whitehead link 5_1^2.

$$\pi_1(S^3 \setminus 5_1^2) = \langle f, g | (f^{-1}gfg^{-1})(fgf^{-1}g^{-1})(fg^{-1}f^{-1}g)(f^{-1}g^{-1}fg) = 1 \rangle.$$



The faithful representation in $\mathrm{PSL}(2, \mathbb{C})$ is given by $f \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, g \to \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$.

Thus, $\pi_1(S^3 \setminus 5_1^2) < \mathrm{PSL}(2, \mathbb{Z}[\sqrt{-1}])$.

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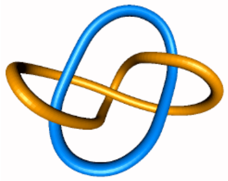

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[Sato, 2001] $\mathcal{J}(\pi_1(S^3 \setminus 5_1^2)) = 2$.

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[Callahan, 2009] The figure-eight knot group is the only torsion-free Jørgensen group.

[Callahan, 2009] If $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_z = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$ generate a group of 2-bridge knot or link, then in the case of a knot:

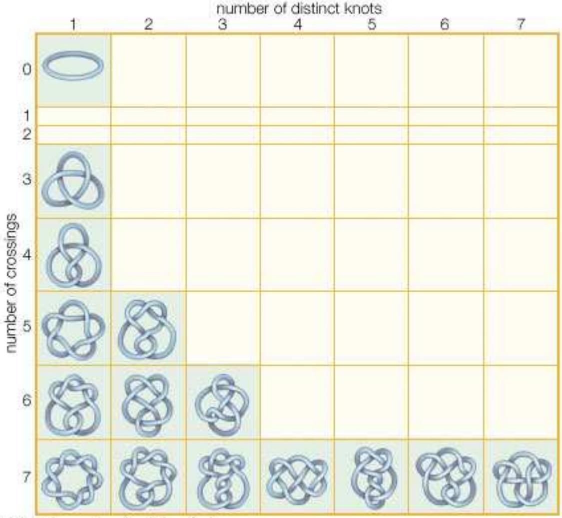
$$1 < \mathcal{J}(\langle f, g_z \rangle) \leq |z| < 4,$$

and in the case of a link:

$$1 < \mathcal{J}(\langle f, g_z \rangle) \leq |z|^2 < 16.$$

[Hoste, Shanahan, 2001] Polynomials for z in the case of 2-bridge twisted knots.

The initial table of knots.



If $G = \pi_1(S^3 \setminus K)$ is 2-generated then

$$\mathcal{J}(K) := \inf_{\langle f,g \rangle = G} \mathcal{J}(f,g)$$

Jørgensen number as hyperbolic knot invariant.

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[Callahan, 2009] For few 2-bridge knots:

$\lvert z\rvert$	polynomial
$\mathcal{J}(5_2) = 1.32471796\ldots$	$1 - 2z + z^2 - z^3$
$\mathcal{J}(6_1) = 1.55603019\ldots$	$1 - 2z + 3z^2 - z^3 + z^4$
$\mathcal{J}(7_4) = 2.20556943\ldots$	$1 + 4z - 4z^2 + z^3$
$\mathcal{J}(7_7) = 1.55603019\ldots$	$1 - z + 3z^2 - 2z^3 + z^4$

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If $G = \pi_1(S^3 \setminus K)$ is not 2-generated then take all 2-generated non-elementary subgroups:

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Behavior of Jørgensen numbers: from orbifold groups to a knot group.

Figure-eight orbifolds.

Let $4_1(n)$, $n \geq 4$, be a 3-orbifold with singular set the figure-eight knot 4_1 and singular group $\mathbb{Z}_n$. It is known that $4_1(n)$ is hyperbolic.

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The figure-eight orbifold group:

$$\pi^{\mathrm{orb}} 4_1(n) = \langle \; f_n, g_n \mid f_n^n = g_n^n = 1, \quad f_n \, [g_n, f_n] \, g_n \, [g_n, f_n]^{-1} = 1 \; \rangle.$$

These is a faithful representation in $\mathrm{PSL}(2, \mathbb{C})$ given by

$$f_n \to \begin{pmatrix} \cos \frac{\pi}{n} & i e^{\rho_n/2} \sin \frac{\pi}{n} \\ i e^{-\rho_n/2} \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}, \qquad g_n \to \begin{pmatrix} \cos \frac{\pi}{n} & i e^{-\rho_n/2} \sin \frac{\pi}{n} \\ i e^{\rho_n/2} \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}$$

where ρ_n is complex distance between axes of f_n and g_n,

$$\cosh \rho_n = \frac{1}{4} \left(1 + \mathrm{ctg}^2(\pi/n) - i \sqrt{3 \, \mathrm{ctg}^4(\pi/n) + 14 \, \mathrm{ctg}^2(\pi/n) - 5} \right).$$

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For above generators we have:

$$\mathcal{J}(f_n, g_n) = 4\sin^2(\pi/n) + \sqrt{1 + 4\sin^2(\pi/n)}.$$

Convergence of Jørgensen numbers. The figure-eight knot case.

Thm. Let $n \geq 4$. Then the following inequalities hold:

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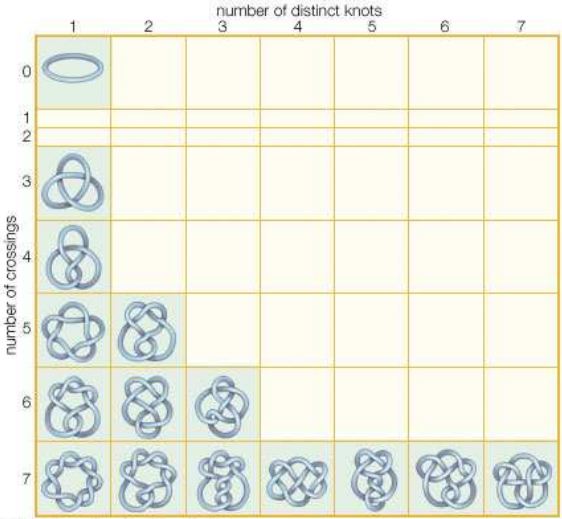
Corollary.
The following convergence holds:

$$\lim_{n\to\infty} \mathcal{J}(\pi^{\mathrm{orb}} 4_1(n)) = \mathcal{J}(\pi_1(S^3 \setminus 4_1)).$$

The initial table of knots.



Consider the knot 5_2. Recall that

$$\pi_1(S^3 \setminus 5_2) = \langle f, g \quad | (f^{-1}g^{-1}fgf^{-1}g^{-1}) \cdot f = g \cdot (f^{-1}g^{-1}fgf^{-1}g^{-1}) \rangle.$$

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There is a faithful representation in $\mathrm{PSL}(2, \mathbb{C})$ given by $f \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g \to \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$, where

$$z = \frac{1}{3} - \frac{(1 - i\sqrt{3}) \cdot \sqrt[3]{11 + 3\sqrt{69}}}{6\sqrt[3]{2}} + \frac{5 \cdot (1 + i\sqrt{3})}{3\sqrt[3]{4}\sqrt[3]{11 + 3\sqrt{69}}} = 0.21508... + i \cdot 1.30714...$$

is a root of $1 - 2z + z^2 - z^3$.

Consider the knot 5_2. Recall that

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[Callahan, 2009] $\mathcal{J}(\pi_1(S^3 \setminus 5_2)) = \mathcal{J}(f, f^{-1}g^{-1}fgf^{-1}g^{-1}) =| z |= 1.324717957\ldots$

The knot 5_2 and related orbifolds.

Let $5_2(n)$, $n \geq 3$, be a 3-orbifold with singular set the knot 5_2 and singular group $\mathbb{Z}_n$. It is known that $5_2(n)$ is hyperbolic.

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$$\pi^{\mathrm{orb}} 5_2(n) = \langle f_n, g_n \mid f_n^n = g_n^n = f_n \cdot (g_n f_n g_n^{-1} f_n^{-1} g_n f_n) \cdot g_n^{-1} \cdot (g_n f_n g_n^{-1} f_n^{-1} g_n f_n)^{-1} = 1 \rangle$$

has a faithful representation in $\mathrm{PSL}(2, \mathbb{C})$ given by

$$f_n = \begin{pmatrix} \cos(\pi/n) & ie^{\rho_n/2} \sin(\pi/n) \\ ie^{-\rho_n/2} \sin(\pi/n) & \cos(\pi/n) \end{pmatrix}$$

and

$$g_n = \begin{pmatrix} \cos(\pi/n) & ie^{-\rho_n/2} \sin(\pi/n) \\ ie^{\rho_n/2} \sin(\pi/n) & \cos(\pi/n) \end{pmatrix},$$

where ρ_n is complex distance between axes of f_n and g_n. Thus, $\cosh(\rho_n)$ satisfies the equation

$$8 \cdot x^3 - 4 \cdot (N^2 - 1) \cdot x^2 + (4N^4 + 8N^2 - 4) \cdot x - N^6 + N^4 + 9N^2 - 1 = 0$$

with $N = \mathrm{ctg}(\pi/n)$.

Convergence of Jørgensen numbers. The knot 5_2 case.

Prop.
The following convergence of the upper bound of $\mathcal{J}(\pi^{\mathrm{orb}} 5_2(n))$ holds:
$\mathcal{J}(g_n^{-1}, g_n f_n g_n^{-1} f_n^{-1} g_n f_n) \to \mathcal{J}\big(\pi_1(S^3 \setminus 5_2)\big)$ for $n \to \infty$.

Necessary discreteness conditions analogues to Jørgensen's.

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Further directions.

[Tan, 1989] Suppose that $f, g \in \mathrm{PSL}(2, \mathbb{C})$ generate a discrete group. If $\mathrm{tr}^2(f) \neq 1$, then the following inequality holds: $|\mathrm{tr}^2(f) - 1| + |\mathrm{tr}[f, g]| \geq 1$.

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For $f, g \in \mathrm{PSL}(2, \mathbb{C})$ such that $\mathrm{tr}[f, g] \neq 0$ define

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Two-generated discrete group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be *Tan group* if it can be generated by f and g such that $\mathcal{T}(f,g) = 1$.

Parabolic and elliptic cases.

Prop. Let 4_1 be the figure-eight knot. Then

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Prop. Let $4_1(n)$ be the figure-eight orbifold, $n \ge 4$. Then

$$1 \le \mathcal{T}(\pi^{\mathrm{orb}} 4_1(n)) \le \begin{cases} 3-4\sin^2(\pi/n)+\sqrt{3-4\sin^2(\pi/n)}, & n \le 7, \\ 1+\sqrt{2}+\sqrt{1+\sqrt{2}}, & n = 8, \\ \sqrt{7-8\sin^2(\pi/n)}+\sqrt{3-4\sin^2(\pi/n)}, & n \ge 9. \end{cases}$$

Tan numbers.

Problem: Find Tan numbers for knot groups.

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Problem: Find all Tan groups.

Gehring – Martin – Tan numbers.

[Gehring – Martin, 1989; Tan, 1989]
Suppose that $f, g \in \mathrm{PSL}(2, \mathbb{C})$ generate a discrete group. If $\mathrm{tr}[f, g] \neq 1$ then the following inequality holds: $|\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f, g] - 1| \geq 1$.

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Two-generated discrete group $G < \mathrm{PSL}(2, \mathbb{C})$ is said to be *GMT-extreme* if it can be generated by f and g such that $\mathcal{G}(f, g) = 1$.

GMT-number of the figure-eight knot group.

Thm. For the figure-eight knot group we have $\mathcal{G}\big(\pi_1(S^3 \setminus 4_1)\big) = 3$. Thus, it is not GMT-extreme.

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Suppose that f is elliptic of order n, so $\mathrm{tr}^2(f) = 4\cos^2(k\pi/n)$.
If $n=2$ then $\mathcal{G}(f,g) > 2$,
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The figure-eight ofbifold group which is GMT-extreme.

Thm. Let $n \geq 4$. Then for the figure-eight orbifold groups the following inequalities hold:

$$1 \leq \mathcal{G}(\pi^{\mathrm{orb}} 4_1(n)) \leq 3 - 4\sin^2(\pi/n).$$

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Corollary. The figure-eight orbifold group $\pi^{\mathrm{orb}} 4_1(4)$ is GMT-extreme.

Problem: Find all Gehring — Martin — Tan (GMT-extreme) groups.

Thank you!