On simplicial resolutions of framed links

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Contents

1 background

2 framed links

3 main result

A Δ-set means a sequence of sets $X = \{X_n\}_{n\geq 0}$ with faces $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, s.t.

$$d_i d_j = d_j d_{i+1}$$

for $i \geq j$, which is called the Δ-identity.

A Δ-set $G = \{G_n\}_{n\geq 0}$ is called a Δ-group if each G_n is a group, and each face d_i is a group homomorphism.

A simplicial set means a sequence of sets $X = \{X_n\}_{n\geq 0}$ with faces $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, and degeneracies $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$, such that

1. $$d_j d_i = d_{i-1} d_j \quad \text{for} \quad j < i$$
2. $$s_j s_i = s_{i+1} s_j \quad \text{for} \quad j \geq i$$
3. $$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \mathrm{id} & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases}$$

A simplicial group $G = \{G_n\}_{n\geq 0}$ is a simplicial set together with each G_n is a group, and each face d_j, each degeneracy s_i is a group homomorphism.

The loop spaces

$$
\begin{aligned}
\Omega X &= \text{loop space of } X \\
&= \{(I, \partial I) \longrightarrow (X, *)\}
\end{aligned}
$$

The loop spaces are a class of important mapping spaces. The study on the various loop spaces has long been a hot topic in algebraic topology and related areas.

Combinatorial models for mapping spaces

James construction $J(X)$

Milnor's construction $F[X]$

Kan's construction $\Omega(-)$

Carlsson free product

For some space X, try to find a simplicial group G, s.t. $|G| \simeq \Omega X$.

For S^3, it is known that ΩS^3 has a good model $J(S^2)$, i.e., $J(S^2) \simeq \Omega S^3$.

Our Work :

Construct a link group model for ΩS^3.

Simplicial/Δ-structures on Braids

- A lot of work have been done on simplicial/Δ-structures of braid groups and mapping class groups.
- Basic idea: Intuitively think $(n+1)$-strand braids as n-simplices.
  Faces = remove strands.
  Degeneracies = cabling strands.

Simplicial structures on framed links

Idea: Naive cablings on framed links.

A framed link, denoted by L^f, is a smooth link L in S^3 equipped with a smooth nonzero normal vector field f defined on a small neighborhood of L.

A framed link L determines a 4-manifold M_L obtained by adding 2-handles to the 4-ball B^4 along the circles in L using their framings.

- Lickorish and Wallace showed that any orientable 3-manifold is ∂M_L for some framed link L.
- Kirby described two operations (the calculus) on a framed link and prove that $\partial M_L = \partial M_{L'}$, if and only if we can pass from L to $L^{'}$ by a sequence of these operations.
- Reshetikhin and Turaev constructed invariants of 3-manifolds via framed link polynomials and quantum groups.

A twisted-annulus link in S^3 is a finite collection of disjoint images of smooth embeddings of line bundles over the circle in S^3, equipped with an orientation.

For each twisted-annulus link, there is an associated framed link given by the collection of core curves framed by the positive unit normal vector field of the twisted-annulus.

Conversely, for any framed link L^f, there is a twisted-annulus link consisting of a collection of annuli A obtained by pushing the link L along the direction X given by the cross product of the framing f with the tangent vector field on L so that $f/|f|$ becomes an orientation of A. The resulting twisted-annulus link is denoted by $(L; X, A)$ with the exponential map

$$\exp\colon \{(P, tX(P)) \mid -1 < t < 1, P \in L\} \longrightarrow A$$

by projecting the normal vector fields in the direction of X to the annuli A with $\exp(P, 0) = P$ for $P \in L$.

For each $k \geq 0$, let

$$L_k = \exp\left\{ \left(P, \left(1 - \frac{1}{k+1}\right) X(P)\right) \;\middle|\; P \in L \right\}.$$

Note that $L_0 = L$ and L_k is a copy of L being pushed away from L by the vector field X.

The *naive cabling* on the twisted-annulus link $(L; X, A)$ is a sequence of links $\mathbb{L}_n$, $0 \leq n < \infty$, given by

$$\mathbb{L}_n = \{L_0, L_1, \ldots, L_n\}.$$

(**Note.** If L is a framed q-link, then $\mathbb{L}_n$ is framed $(n+1)q$-link.)

link group $G(\mathbb{L}_n) = \pi_1(S^3 \smallsetminus \mathbb{L}_n)$

simplicial group structure on $\mathbb{G}(L; X) = \{G(\mathbb{L}_n)\}_{n \geq 0}$

$$\begin{array}{lc}
n = 0 & G(\mathbb{L}_0) = \pi_1(S^3 \backslash \mathbb{L}_0) \\
 & d_0 \uparrow\, d_1 \uparrow \quad \downarrow s_0 \\
n = 1 & G(\mathbb{L}_1) = \pi_1(S^3 \backslash \mathbb{L}_1) \\
 & d_0 \uparrow\, d_1 \uparrow\, d_2 \uparrow \quad \downarrow s_0 \,\downarrow s_1 \\
n = 2 & G(\mathbb{L}_2) = \pi_1(S^3 \backslash \mathbb{L}_2)
\end{array}$$

simplicial group structure on $\mathbb{G}(L, X)$

- the i-th face homomorphism: removing the i-th copy of L
- the i-th degeneracy homomorphism: doubling the i-th copy of L

faces

$$d_i\colon G(\mathbb{L}_n) \to G(\mathbb{L}_{n-1})$$

is induced by

$$S^3 \smallsetminus \mathbb{L}_n \hookrightarrow S^3 \smallsetminus \tilde{d}_i\mathbb{L}_n \xrightarrow{\phi_i} S^3 \smallsetminus \mathbb{L}_{n-1},$$

where $\tilde{d}_i\mathbb{L}_n = \{L_0, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n\}$ and ϕ_i is the homeomorphism induced by the flow associated with the vector field X that pushes the links $L_{i+1}, L_{i+2}, \ldots, L_n$ to $L_i, L_{i+1}, \ldots, L_{n-1}$ in the annuli A, respectively, with fixing $L_0, \ldots, L_{i-1}$.

degeneracies

Let $\tau(i) = \left(1 - \frac{1}{i+1}\right)$, and A_i the annuli bounded by L_i and the middle link

$$L_{i+1/2} = \exp\{(P, (\frac{1}{2}\tau(i) + \frac{1}{2}\tau(i+1))X(P)) \mid P \in L\}$$

between L_i and L_{i+1} in A.

degeneracies

$$s_i\colon G(\mathbb{L}_n) \to G(\mathbb{L}_{n+1})$$

is induced by

$$S^3 \smallsetminus \mathbb{L}_n \overset{\iota}{\hookleftarrow} S^3 \smallsetminus (\mathbb{L}_n \cup A_i) \hookrightarrow S^3 \smallsetminus (\mathbb{L}_n \cup L_{i+1/2}) \xrightarrow{\psi_i} S^3 \smallsetminus \mathbb{L}_{n+1}.$$

The first map in the composition is a homotopy inverse r of ι. Since $S^3 \smallsetminus (\mathbb{L}_n \cup A_i)$ is a strong deformation retract of $S^3 \smallsetminus \mathbb{L}_n$, $\iota_*\colon \pi_1(S^3 \smallsetminus (\mathbb{L}_n \cup A_i)) \longrightarrow \pi_1(S^3 \smallsetminus \mathbb{L}_n)$ is an isomorphism and $r_* = \iota_*^{-1}$. ψ_i is the homeomorphism similar to ϕ_i, where the flow pushes $L_{i+1/2}, L_{i+1}, \ldots, L_n$ to $L_{i+1}, L_{i+2}, \ldots, L_{n+1}$, respectively.

$\mathbb{G}(L; X)$ is a simplicial group

Any link L has a unique splitting decomposition

$$L \cong L^{[1]} \sqcup L^{[2]} \sqcup \cdots \sqcup L^{[p]},$$

where each $L^{[j]}$ is a nonsplittable sublink of L and $L^{[1]}, \ldots, L^{[p]}$ are mutually separated by embedded 2-spheres.

For a framed link L^f, each nonsplittable factor $L^{[j]}$ has the canonical frame $f^{[j]}$ given by the restriction $f|_{L^{[j]}}$. Thus $(L; X, A)$ has

$$(L; X, A) \cong (L^{[1]}; X^{[1]}, A^{[1]}) \sqcup (L^{[2]}; X^{[2]}, A^{[2]}) \sqcup \cdots \sqcup (L^{[p]}; X^{[p]}, A^{[p]}).$$

Main Result

Theorem (Lei-L-Wu)

L^f : a framed link in S^3

$(L; X, A)$: the associated twisted-annulus link

$(L; X, A) \cong (L^{[1]}; X^{[1]}, A^{[1]}) \sqcup (L^{[2]}; X^{[2]}, A^{[2]}) \sqcup \cdots \sqcup (L^{[p]}; X^{[p]}, A^{[p]})$

s.t.

$(L^{[i]}; X^{[i]}, A^{[i]}), i = 1, \cdots, k$: nontrivial nonsplittable twisted-annulus links,

$(L^{[j]}; X^{[j]}, A^{[j]}), j = k+1, \cdots, p$: trivial twisted-annulus knots,

Then

$$|\mathbb{G}(L; X)| \simeq \Omega(\bigvee^{k} S^3).$$

Ideas of Proof

Given a simplicial group $G = \mathbb{G}(L; X)$, we try to show that

$$G \simeq \Omega(\bigvee^{k} S^3)$$

We need to show that the classifying space $BG \simeq \bigvee^{k} S^3$.

1. Step 1. $\pi_1(BG) = \pi_0(G) = \{1\}$.
2. Step 2. Compute the homology $H_*(BG)$ is the same as $H_*(\bigvee^{k} S^3)$.

Thank you !