Homotopy groups, braids and links

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Definition of Homotopy Groups

Let S^n be the n-dimensional sphere with a basepoint (say North Pole N). Let X be a space with a given basepoint b. The **n-th homotopy group** $\pi_n(X)$ is defined to be the set of the homotopy classes of all continuous maps $f\colon S^n \to X$ with $f(N) = b$.

- If $n = 0$, $\pi_0(X)$ is one-to-one correspondent to the set of path-connected components of X. $\pi_0(X)$ is not a group in general.
- If $n = 1$, $\pi_1(X)$ is the **fundamental group** of the space X.
- The (higher) homotopy groups $\pi_n(X)$ with $n \geq 2$ are abelian groups.

★ The homotopy groups is a cornerstone of homotopy theory.

History on Homotopy Groups

- In the late 19th century **Camille Jordan** introduced the notion of homotopy and used the notion of a homotopy group, without using the language of group theory.
- A more rigorous approach was adopted by **Henri Poincaré** in his 1895 set of papers Analysis situs where the related concepts of homology and the fundamental group were also introduced.
- Higher homotopy groups were first defined by **Eduard Čech** in 1932. (His first paper was withdrawn on the advice of **Pavel Sergeyevich Alexandrov** and **Heinz Hopf**, on the grounds that the groups were commutative so could not be the right generalizations of the fundamental group.)
- **Witold Hurewicz** is also credited with the introduction of homotopy groups in his 1935 paper and also for the **Hurewicz theorem** which can be used to calculate some of the groups.

History on Homotopy Groups

- It was originally conjectured that the homotopy groups of spheres are the same as the homology. Then Heinz Hopf invented famous **Hopf map** $\eta\colon S^3 \to S^2$ in 1931, which gives a generator for $\pi_3(S^2) = \mathbb{Z}$.
- In 1938 **Lev Pontrjagin** made a computational mistake for stating that $\pi_{n+1}(S^n) = 0$ for $n \geq 3$. However his method was posing the basic problem of cobordism theory, by establishing an isomorphism between homotopy groups and the group of cobordism classes of framed manifolds.
- In 1954, the Pontrjagin isomorphism was generalized by **René Thom** with an application to give the classifications of manifolds up to cobordism. Thom received a Fields Medal because of this work.

Some Higher Homotopy Groups of S^2

$$\pi_n(S^2) = \begin{cases} 0 & \text{if} \quad n = 0, 1 \\ \mathbb{Z} & \text{if} \quad n = 2, \\ \mathbb{Z} & \text{if} \quad n = 3, \\ \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n = 4, 5, 7, 8, 11 \\ \mathbb{Z}/12\mathbb{Z} & \text{if} \quad n = 6, \\ \mathbb{Z}/3\mathbb{Z} & \text{if} \quad n = 9, \\ \mathbb{Z}/15\mathbb{Z} & \text{if} \quad n = 10, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if} \quad n = 12, 15 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if} \quad n = 13, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/84\mathbb{Z} & \text{if} \quad n = 14, \\ \mathbb{Z}/6\mathbb{Z} & \text{if} \quad n = 16, \\ \mathbb{Z}/30\mathbb{Z} & \text{if} \quad n = 17, 18, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & \text{if} \quad n = 19, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \text{if} \quad n = 20, 21, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/132\mathbb{Z} & \text{if} \quad n = 22. \end{cases}$$

Methods for computing homotopy groups

The determination of the general homotopy groups $\pi_*(S^n)$ is a **fundamental unsolved problem** in algebraic topology.

- *EHP* sequences.
- Homological methods (spectral sequences): Adams spectral sequences, Adams-Novikov spectral sequences $\cdots\cdots$

Apart from computations,

- Elements in homotopy groups should have particular meanings: For instance, Hopf map $S^3 \to S^2$ is a generators for $\pi_3(S^2) = \mathbb{Z}$.
- CMN theorem gives a nice description for the groups $\pi_*(S^{2n+1})$.

Our Interest on Homotopy Groups

- **Uniform understanding** of homotopy groups is possibly more important than computation.
- **Interactions** between homotopy groups and other areas of mathematics.

Our current progress on the interactions between homotopy groups and

- braid groups (Brunnian braids). [F. Cohen, J. Berrick, Yan Loi Wong, V. Vershinin, V. Bardakov, R. Mikhailov, Jingyan Li, W.]
- mapping class groups. [Berrick, Liz Hanbury, W.]
- link groups. [Fuquan Fang, Fengchun Lei, Yu Zhang, Fengling Li, W.]
- Vassiliev invariants. [F. Cohen, Jingyan Li, V. Vershinin, W.]

Combinatorial description of $\pi_*(S^2)$

- Let
$$F_n = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n \rangle$$
be the one-relator group generated by $x_0, \ldots, x_n$ with the defining relation $x_0 \cdots x_n = 1$. rank n with a basis given by $\{x_1, \ldots, x_n\}$.)
- Let $R_i = \langle x_i \rangle^{F_n}$ be the normal closure of x_i in F_n for $0 \leq i \leq n$. We can form a symmetric commutator subgroup
$$[R_0, R_1, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_{n+1}} [\ldots [R_{\sigma(0)}, R_{\sigma(1)}], \ldots, R_{\sigma(n)}],$$
- **Theorem (Wu, 1994, published version 2001).** For $n \geq 1$, there is an isomorphism
$$\pi_{n+1}(S^2) \cong \frac{R_0 \cap \cdots \cap R_n}{[R_0, \ldots, R_n]_S}$$
This quotient group is isomorphic to the center of the group $F_n/[R_0, R_1, \ldots, R_n]_S$.

van Kampen Type Theorem for Higher Homotopy Groups

- The Seifert-van Kampen theorem a basic tool for computing the fundamental group, there is no simple way to calculate the homotopy groups of a space by breaking it up into smaller spaces.
- Some methods developed by **R. Brown and J.-L. Loday** in the 1980s involving a van Kampen type theorem for higher homotopy groups (π_2 and π_3).
- Their results were generalized by **Ellis-Steiner** in the 1980s with only properly advertised in the recent paper of **G. Ellis and R. Mikhailov** (Advances in Math. 2010).
- Ellis-Mikhailov's paper, as they stated, generalized Brown-Loday theorem and my result.

Braid group actions

- There is an action of the braid group B_{n+1} on $F_n = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n \rangle$ by the Artin representation, which induces an action of B_{n+1} on the quotient group $F_n/[R_0, R_1, \ldots, R_n]_S$.

- **Theorem (Wu, 2002)** The center of $F_n/[R_0, R_1, \ldots, R_n]_S$ is exactly given by the fixed set of the pure braid group P_{n+1} action on $F_n/[R_0, R_1, \ldots, R_n]_S$ for $n \geq 3$.

Homotopy Groups and Brunnian braids

Let $\mathrm{Brun}_n(M)$ be the group of Brunnian braids on the surface M. Then inclusion D^2 into S^2 by regarding D^2 as the upper hemisphere induces a group homomorphism $f_*\colon\ \mathrm{Brun}_n(D^2) \to \mathrm{Brun}_n(S^2)$.

- **Theorem (Berrick-Cohen-Wong-W., 2006):** For $n \geq 5$, there is an exact sequence of groups

$$\mathrm{Brun}_{n+1}(S^2) \hookrightarrow \mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

Roughly speaking $\pi_{n-1}(S^2)$ is given by the n-strand Brunnian braids on S^2 modulo the n-strand Bruunian on D^2.

Brunnian Braids on General Surfaces

Let M be a connected 2-manifold and let $n \geq 4$. Let

$$R_n(M) = \prod_{\sigma \in \Sigma_{n-1}} [[\langle\langle A_{\sigma(1),n}\rangle\rangle^P, \langle\langle A_{\sigma(2),n}\rangle\rangle^P], \ldots, \langle\langle A_{\sigma(n-1),n}\rangle\rangle^P]$$

be the symmetric commutator subgroup, where $\langle\langle A_{i,j}\rangle\rangle^P$ is the normal closure of the braid $A_{i,j}$ in $P_n(M)$.

Theorem (Bardakov-Mikhailov-Vershinin-W., 2012):

1. If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then
$$\mathrm{Brun}_n(M) = R_n(M).$$
2. If $M = S^2$ and $n \geq 5$, then there is a short exact sequence
$$R_n(S^2) \hookrightarrow \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$
3. If $M = \mathbb{R}\mathrm{P}^2$, then there is a short exact sequence
$$R_n(\mathbb{R}\mathrm{P}^2) \hookrightarrow \mathrm{Brun}_n(\mathbb{R}\mathrm{P}^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

Homotopy groups and Mirror Reflections on Braids

Let $\chi\colon B_n \to B_n$ be the mirror reflection, namely algebraically χ is an endomorphism with $\chi(\sigma_i) = \sigma_i^{-1}$. Let Bd_n be the (normal) subgroup of B_n consisting of boundary Brunnian braids. (Roughly speaking $\mathrm{Bd}_n = \partial(\mathrm{Brun}_{n+1}(D^2))$ for certain homomorphism $\partial\colon P_{n+1} \to P_n$.

- **Theorem (Jingyan Li and W., 2009):** There is an isomorphism of groups

  $$\mathrm{Fix}^{\chi}(B_n/\mathrm{Bd}_n) \cong \pi_n(S^2)$$

  for $n \geq 3$.

Namely $\pi_n(S^2)$ is given as the fixed-point set of the mirror reflection on the quotient group B_n/Bd_n.

Question on $\pi_*(S^k)$

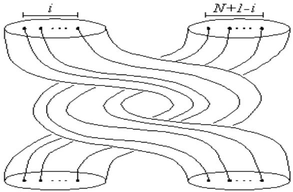
- It has been the concern of many people whether one can give a combinatorial description of the homotopy groups of higher dimensional spheres, ever since a description of $\pi_*(S^2)$ was announced in 1994.

our construction, Mikhailov-Wu

- We give a combinatorial description of $\pi_*(S^k)$ for any $k \geq 3$ by using the free product with amalgamation of pure braid groups.
- Our construction is as follows. Given $k \geq 3,\ n \geq 2$, let P_n be the n-strand Artin pure braid group with the standard generators $A_{i,j}$ for $1 \leq i < j \leq n$. We construct a (free) subgroup $Q_{n,k}$ of P_n from cabling as follows.
- Our cabling process starts from $P_2 = \mathbb{Z}$ generated by the 2-strand pure braid $A_{1,2}$.

The construction of $Q_{n,k}$: Step 1

Consider the 2-strand pure braid $A_{1,2}$. Let ξ_i be $(k-1)$-strand braid obtained by inserting i parallel strands into the tubular neighborhood of the first strand of $A_{1,2}$ and $k-i-1$ parallel strands into the tubular neighborhood of the second strand of $A_{1,2}$ for $1 \leq i \leq k-2$. [From Cohen-Wu 2004, 2011]



Where N+1=k-1

The construction of $Q_{n,k}$: Step 2

- Let $\alpha_k = [\ldots[[\xi_1^{-1}, \xi_1\xi_2^{-1}], \xi_2\xi_3^{-1}], \ldots, \xi_{k-3}\xi_{k-2}^{-1}, \xi_{k-2}]$ be a fixed choice of $(k-1)$-strand braid, which is a nontrivial $(k-1)$-strand Brunnian braid.

- For a group G and $g, h \in G$, we use the notation $[g, h] := g^{-1}h^{-1}gh$.

The construction of $Q_{n,k}$: Step 3

- By applying the cabling process as in Step 1 to the element α_k, we insert parallel strands into the tubular neighborhood of the strands of α_k in any possible way to obtain n-strand braids. As the order in which the strands are inserted is arbitrary, there are $\binom{n-1}{k-2}$ ways of doing this. Label the $\binom{n-1}{k-2}$ n-strand braids obtained in this way by y_j for $1 \leq j \leq \binom{n-1}{k-2}$.

- **It is too difficult to draw a picture for y_j now!**

- Let $Q_{n,k}$ be the subgroup of P_n generated by y_j for $1 \leq j \leq \binom{n-1}{k-2}$.

Free Product of Pure Braid Groups with Amalgamation

Now consider the free product with amalgamation

$$P_n *_{Q_{n,k}} P_n.$$

Namely this amalgamation is obtained by identifying the elements y_j in two copies of P_n. Let $A_{i,j}$ be the generators for the first copy of P_n and let $A'_{i,j}$ denote the generators $A_{i,j}$ for the second copy of P_n. Let $R_{i,j} = \langle A_{i,j}, A'_{i,j} \rangle^{P_n *_{Q_{n,k}} P_n}$ be the normal closure of $A_{i,j}, A'_{i,j}$ in $P_n *_{Q_{n,k}} P_n$. Let

$$[R_{i,j} \mid 1 \leq i < j \leq n]_S = \prod_{\{1,2,\ldots,n\}=\{i_1,j_1,\ldots,i_t,j_t\}} [[R_{i_1,j_1}, R_{i_2,j_2}], \ldots, R_{i_t,j_t}]$$

be the product of all commutator subgroups such that each integer $1 \leq j \leq n$ appears as one of indices at least once.

Result of Mikhailov-W., 2013

Let $k \geq 3$. The homotopy group $\pi_n(S^k)$ is isomorphic to the center of the group

$$(P_n *_{Q_{n,k}} P_n)/[R_{i,j} \mid 1 \leq i < j \leq n]_S$$

for any n if $k > 3$ and any $n \neq 3$ if $k = 3$.

- **Note.** The only exceptional case is $k = 3$ and $n = 3$. In this case, $\pi_3(S^3) = \mathbb{Z}$ while the center of the group is $\mathbb{Z}^{\oplus 4}$.

Link Groups and Loop Spaces

This is a joint work with Fengchun Lei and Fengling Li.

See Fengling's talk.

One can get simplicial group models for the loops on wedge of 3-spheres from the link groups of naive cablings on framed links.

Questions

- **Serre Problem:** Determine the rank of $\pi_n(S^3)$, that is the number of minimal generators for $\pi_n(S^3)$. Try to study the upper bounds and lower bounds of the rank of $\pi_n(S^3)$.
- **Cohen Problem:** Determine the series

$$f(x) = \sum_{n=4}^{\infty} \frac{|\pi_n(S^3)|}{n!} x^n$$

as a function, where $|\pi_n(S^3)|$ is the order of the group $\pi_n(S^3)$.

Thank You!