On the subgroups of the groups of Brunnian links

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The purpose of this talk aims to the intersecting subgroup

$$R_1 \cap R_2 \cap \cdots \cap R_n$$

of a Brunnian link group $G(L_n)$, where R_i is the normal closure of the i th meridian.

The answer will be given as an applicantion of the Ellis-Mikhailov Theorem to link groups.

Connectivity Hypothesis

Let G be a group. An m-tuple of normal subgroups $(R_1, \ldots, R_m)$ of G is called *connected* if either

1. $m \leq 2$ or
2. $m \geq 3$ with the property that: for all subsets $I, J \subseteq \{1, \cdots, m\}$ with $|I| \geq 2, |J| \geq 1$

$$\left(\bigcap_{i \in I} R_i\right) \cdot \prod_{j \in J} R_j = \bigcap_{i \in I} \left(R_i \cdot \prod_{j \in J} R_j\right). \qquad (1)$$

Homotopy Colimit

Let G be a group with normal subgroups $R_1, \ldots, R_n$. Let $X(G; R_1, \ldots, R_n)$ be the homotopy colimit of the cubical diagram obtained from classifying spaces $B(G/\prod_{i \in I} R_i)$ with the maps

$$B(G/\prod_{i \in I} R_i) \to B(G/\prod_{i' \in I'} R_{i'})$$

induced by the canonical quotient homomorphism $G/\prod_{i \in I} R_i \twoheadrightarrow G/\prod_{i' \in I'} R_{i'}$ for $I \subseteq I'$, where I ranges over all proper subsets $I \subsetneq \{1, \ldots, n\}$.

Ellis-Mikhailov Theorem, 2010

In the following theorem, the notation $(\cdots \hat{a} \cdots)$ means that the letter a is removed.

- **Ellis-Mikhailov Theorem** Let G be a group with normal subgroups $R_1, \ldots, R_n$ with $n \geq 2$. Let $X = X(G; R_1, \ldots, R_n)$. Suppose that the $(n-1)$-tuple

$$(R_1, \ldots, \hat{R}_i, \ldots, R_n)$$

is connected for each $1 \leq i \leq n$. Then

$$\pi_n(X) \cong \frac{R_1 \cap \cdots \cap R_n}{\prod_{I \cup J = \{1,\ldots,n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]}.$$

Symmetric Commutator Subgroups

Let G be a group and let $R_1, \ldots, R_n$ be subgroups of G. The *symmetric commutator subgroup* $[[R_1, R_2], \ldots, R_n]_S$ defined by

$$[[R_1, R_2], \ldots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where $[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]$ is the subgroup generated by the left iterated commutators

$$[[[g_1, g_2], g_3], \ldots, g_n]$$

with $g_i \in R_{\sigma(i)}$. For convenience, let $[R_1]_S = R_1$.

Partitions

Let (X, A) be a pair of spaces. An *n-partition* of X relative to A means a sequence of subspaces $(A_1, \dots, A_n)$ of X such that

1. $A = A_i \cap A_j$ for each $1 \leq i < j \leq n$ and
2. $X = \bigcup_{i=1}^{n} A_i$.

An n-partition $(A_1, \dots, A_n)$ of X relative to A is called *cofibrant* if the inclusion

$$\bigcup_{i \in I} A_i \hookrightarrow \bigcup_{j \in J} A_j$$

is a cofibration for any $I \subseteq J \subseteq \{1, 2, \dots, n\}$. Note that for a cofibrant partition, each union $A_I = \bigcup_{i \in I} A_i$ is the homotopy colimit of the diagram given by the inclusions $A_{I'} \hookrightarrow A_{I''}$ for $\emptyset \subseteq I' \subseteq I'' \subsetneq I$.

Li-Wu's result, 2011

Let (X, A) be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant n-partition of X relative to A with $n \geq 2$. Suppose that

i) For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneqq \{1, 2, \ldots, n\}$, the union $\bigcup_{i \in I} A_i$ is a path-connected $K(\pi, 1)$-space.

ii) The inclusion $A \to A_i$ induces an epimorphism of the corresponding fundamental groups for each $1 \leq i \leq n$.

Li-Wu's result, 2011

Let R_i be the kernel of $\pi_1(A) \to \pi_1(A_i)$ for $1 \leq i \leq n$. Then

1. For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneqq \{1, 2, \ldots, n\}$,

$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

2. For any $1 < k \leq n$ and any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, there is an isomorphism of groups

$$\pi_k(X) \cong \left(\bigcap_{s=1}^{k} \left(R_{i_s} \cdot \prod_{j \in J} R_j \right) \right) \Big/ \left([[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S \cdot \prod_{j \in J} R_j \right),$$

where $J = \{1, 2, \ldots, n\} - I$. In particular,

$$\pi_n(X) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S.$$

Applications to Link Groups

Let M be a path-connected 3-manifold and L be a proper m-link in M with $m \geq 2$. Suppose that for any nonempty sub-link L' of L, the link complement $M \smallsetminus |L'|$ is a $K(\pi, 1)$-space. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be any subsets of $\{1, 2, \ldots, m\}$, with $n \geq 2$, such that

(i) $\Lambda_i \neq \emptyset$ for each $1 \leq i \leq n$.
(ii) $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$.
(iii) $\bigcup_{i=1}^{n} \Lambda_i = \{1, 2, \ldots, m\}$.

Let α_j be the jth meridian of the link L and let

$$R_i = \langle\langle \alpha_j \mid j \in \Lambda_i \rangle\rangle$$

be the normal closure of α_j with $j \in \Lambda_i$ in $\pi_1(M \smallsetminus |L|)$. Then

1. For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,
$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$
2. There is an isomorphism of groups
$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(M).$$

Brunnian Links

Let L_n be a Brunnian n-link in S^3. Namely L_n becomes a trivial link after removing any one of its components.

- The $K(\pi, 1)$ hypothesis in Li-Wu's result does NOT hold for Brunnian links because, for any proper sublink L' of L_n, the link complement $S^3 \smallsetminus L'$ is not a $K(\pi, 1)$ space.

Ellis-Mikhailov connectivity hypothesis works for Brunnian links

- **Observation.** The connectivity equation (1) is equivalent to

$$\overline{\bigcap_{i\in I} R_i} = \bigcap_{i\in I} \overline{R_i} \tag{2}$$

in the quotient group $G/\prod_{j\in J} R_j$.

- Let $G = G(L_n)$, the group of a Brunnian link L_n. Let R_i be the normal closure of the i-th meridian. Then $G/\prod_{j\in J} R_j$ is a free group for $|J| \geq 1$. The connectivity hypothesis holds in this case.

Main Theorem 1

Let $G = G(L_n)$, the group of a Brunnian link L_n. Let R_i be the normal closure of the i-th meridian. Then, for each $2 \leq m \leq n$,

$$\bigcap_{i=1}^{m} R_i = [[R_1, R_2], \ldots, R_m]_S.$$

Outline of Proof, where homotopy theory has to be used

According to Ellis-Mikhailov, we have to control the homotopy colimit of the cubical diagram obtained from the classifying spaces $B(G/\prod_{i\in I} R_i)$.

- In this case, $BG = S^3 \smallsetminus L_n$ and $B(G/\prod_{i\in I} R_i)$ is a wedge of circles for $|I| \geq 1$.

- The determination of $\pi_n(X(G; R_1, \ldots, R_n)) = 0$ is relatively easier because X is simply connected and we can compute the homology of X using iterated Mayer-Vietoris sequences.

Outline of Proof, where homotopy theory has to be used

- This would give a conclusion from Ellis-Mikhailov theorem that

$$R_1 \cap \cdots \cap R_n = \prod_{I \cup J = \{1,\ldots,n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j].$$

- The **tricky part** is to show that particial intersection $\cap_{i \in I} R_i$ is a symmetric commutator subgroup. In this case, the corresponding homotopy colimit is no longer simply connected. We use geometric methods to directly determine the homotopy type of the corresponding homotopy colimit, which is a wedge of circles and higher dimensional spheres.

Second Statement in Ellis-Mikhailov Theorem

Continuation of Ellis-Mikhailov Theorem: There is a left short exact sequence

$$1 \to \pi_{n+1}(X) \to T(R_1, \ldots, R_n) \to \bigcap_{i=1}^{n} R_i.$$

Second Statement in Ellis-Mikhailov Theorem

The group $T(R_1, R_2, \cdots, R_m)$ is generated by symbols

$$a \otimes_{A,B} b$$

where $A \sqcup B = \{1, 2, \cdots, m\}, a \in R_A, b \in R_B$ (here for any $I \subset \{1, 2, \cdots, m\}, R_I = \cap_{i \in I} R_i$), and satisfies the relations

$$a \otimes_{A,B} b = (b \otimes_{B,A} a)^{-1}$$

$$aa' \otimes_{A,B} b = ({}^{a}a' \otimes_{A,B} {}^{a}b)(a \otimes_{A,B} b)$$

$$({}^{u}[u^{-1}, v] \otimes_{U \cup V, W} {}^{u}w)({}^{w}[w^{-1}, u] \otimes_{W \cup U, V} {}^{w}v)({}^{v}[v^{-1}, w] \otimes_{V \cup W, U} {}^{v}u)$$

$$(a \otimes_{A,B} b)(a' \otimes_{A',B'} b')(a \otimes_{A,B} b)^{-1} = {}^{[a,b]}a' \otimes_{A',B'} {}^{[a,b]}b$$

for $A \sqcup B = A' \sqcup B' = \{1, 2, \cdots, m\}, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W = \{1, 2, \cdots, m\}, u \in N_U, v \in N_V, w \in N_W$. Here we define ${}^{x}y = xyx^{-1}$.

Main Theorem 2

Theorem (Presentation of $R_1 \cap R_2 \cap \cdots \cap R_n$). There is a short exact sequence:

$$\bigoplus^{n-1} \mathbb{Z} \rightarrowtail T(R_1, R_2, \cdots, R_n) \twoheadrightarrow \cap_{i=1}^{n} R_i.$$

Outline of Proof and Remarks

- $X = \bigvee^{n-1} S^{n+1}$.

- **Remark.** For $2 \leq m < n$, kernel of $T(R_1, R_2, \cdots, R_m) \twoheadrightarrow \cap_{i=1}^{m} R_i$ is a free abelian group of countably infinite rank.

Thank You !