Brunnian Braids and Lie Algebras

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Brunnian Braid Groups

A braid $\beta \in B_n(M)$ is called **Brunnian** if (1) it is a pure braid and (2) it becomes trivial braid by removing any of its strands. Since the composition of any two Brunnian braids is still Brunnian, the set of Brunnian braids is a normal subgroup of the pure braid group which is denoted by $\mathrm{Brun}_n(M)$. For convenient, $\mathrm{Brun}_n(M)$ is denoted by Brun_n when M is the disc D^2.

Lie Algebra from Descending Central Series of Groups

For a group G, the descending central series

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \geq \Gamma_i(G) \geq \Gamma_{i+1}(G) \geq \cdots$$

is defined by the formulas

$$\Gamma_1(G) = G, \Gamma_{i+1}(G) = [\Gamma_i(G), G] \quad (i \geq 1).$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$:

$$L(G) = \bigoplus_{q=1}^{\infty} \Gamma_q(G)/\Gamma_{q+1}(G).$$

The Relative Lie Algebra $L^P(\mathrm{Brun}_n)$

Since Brun_n is the normal subgroup of the pure braid group P_n, we have the following descending central series

$$\mathrm{Brun}_n = \Gamma_1(P_n)\cap\mathrm{Brun}_n \geq \Gamma_2(P_n)\cap\mathrm{Brun}_n \geq \Gamma_3(P_n)\cap\mathrm{Brun}_n \geq \cdots$$

and **the relative Lie algebra**

$$L^P(\mathrm{Brun}_n) = \bigoplus_{q=1}^{\infty} \Gamma_q(P_n)\cap\mathrm{Brun}_n/\Gamma_{q+1}(P_n)\cap\mathrm{Brun}_n\,.$$

Proposition

Proposition 1

$$L^P(\mathrm{Brun}_n) = \bigcap_{k=1}^{n} \ker(d_k : L(P_n) \to L(P_{n-1})).$$

Remark

The relative Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun}_{\mathrm{n}})$ has better features: (1) it is free generated; (2) it is of finite type; (3) it has connection to the theory of Vassiliev invariants.

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Definition of $\mathcal{K}(n)_k$

We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}\}$.
2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters $A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}$ with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}.$$

Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots[[W', W_1], W_2], \ldots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$ and $W_1, W_2, \ldots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}$.

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Free Generators of Lie Algebra $L^P(\mathrm{Brun}_n)$

Theorem

Theorem 2 The relative Lie algebra $L^P(\mathrm{Brun}_n)$ is a free Lie algebra generated by $\mathcal{K}(n)_1$ as a set of free generators.

Example

Let $n = 4$. The set $\mathcal{K}(4)_1$ is constructed by the following steps:

1) $\mathcal{K}(4)_4 = \{A_{1,4}, A_{2,4}, A_{3,4}\}$.
2) $\mathcal{K}(4)_3 = \{[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \mid 1 \leq j_1, \cdots, j_t \leq 2, \ t \geq 0\}$, where, for $t = 0$, $[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] = A_{3,4}$.
3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}]$. Let

$$\mathrm{ad}^t(b)(a) = [[a, b], \cdots, b]$$

with t entries of b, where $\mathrm{ad}^0(b)(a) = a$. Then W does not contain $A_{2,4}$ if and only if

$$W = \mathrm{ad}^t(A_{1,4})(A_{3,4})$$

for $t \geq 0$.

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3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}]$. Let
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with t entries of b, where $\mathrm{ad}^0(b)(a) = a$. Then W does not contain $A_{2,4}$ if and only if
$$W = \mathrm{ad}^t(A_{1,4})(A_{3,4})$$
for $t \geq 0$.

From the definition, $\mathcal{K}(4)_2$ is given by

$$[[A_{3,4},A_{j_1,4}],\cdots,A_{j_t,4}] \quad \text{and}$$

$$[[[[A_{3,4},A_{j_1,4}],\cdots,A_{j_t,4}],\mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})],\cdots,\mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

where $1\leq j_1,\cdots,j_t\leq 2$ with at least one $j_i=2$, $s_1,\cdots,s_q\geq 0$ and $q\geq 1$.

4) For constructing $\mathcal{K}(4)_1$, let W denote

$$[[[[A_{3,4},A_{j_1,4}],\ldots,A_{j_t,4}],\mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})],\cdots,\mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})]\in\mathcal{K}(4)_2,$$

where, for $q=0$, $W=[[A_{3,4},A_{j_1,4}],\cdots,A_{j_t,4}]$. Then W does not contain $A_{1,4}$ if and only if $q=0$ and $W=[[A_{3,4},A_{2,4}],\cdots,A_{2,4}]$, namely

$$W=\mathrm{ad}^t(A_{2,4})(A_{3,4})$$

for $t\geq 1$.

Thus $\mathcal{K}(4)_1$, which is a set of free generators for $\mathrm{L}^P(\mathrm{Brun}_4)$, is given by

$$W \text{ and } [[W, \mathrm{ad}^{l_1}(A_{2,4})(A_{3,4})], \cdots, \mathrm{ad}^{l_p}(A_{2,4})(A_{3,4})],$$

where $l_i \geq 1$ for $1 \leq i \leq p$ with $p \geq 1$ and

$$W = [[[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

so that each of $A_{2,4}$ and $A_{1,4}$ appears in W at least once.

The Symmetric Bracket Sum of Ideals

Let L be a lie algebra and $I_1, \cdots, I_n$ are its ideals. The fat bracket sum $[[I_1, I_2, \cdots, I_n]]$ of these ideals is defined to be the sub Lie ideal of L generated by all of the commutators

$$\beta^t(a_{i_1}, \cdots, a_{i_t}),$$

where

1) $1 \leq i_s \leq n$;
2) $\{i_1, \cdots, i_t\} = \{1, \cdots, n\}$, that is each integer in $\{1, 2, \cdots, n\}$ appears as at least one of the integers i_s;
3) $a_j \in I_j$;
4) β^t runs over all of the bracket arrangements of weight t (with $t \geq n$).

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3) $a_j \in I_j$;
4) β^t runs over all of the bracket arrangements of weight t (with $t \geq n$).

The symmetric bracket sum of these ideals is defined as

$$[I_1, \ldots, I_l]_S := \sum_{\sigma \in \Sigma_n} [[I_{\sigma(1)}, I_{\sigma(2)}], \cdots, I_{\sigma(n)}],$$

where Σ_n is the symmetric group of degree n.

Lemma

Lemma 3 Let I_j be any Lie ideals of a Lie algebra L with $1 \leq j \leq n$. Then

$$[[I_1, I_2, \cdots, I_n]] = [[I_1, I_2], \cdots, I_n]_S.$$

Let us denote the ideal

$$L[A_{k,n}, [\cdots [A_{k,n}, A_{j_1,n}], \cdots, A_{j_m,n}] \mid j_i \neq k, n;\ j_i \leq n-2, i \leq m;\ m \geq 1]$$

by I_k. Then we have the following theorem.

Theorem

Theorem 4

$$L^P(\mathrm{Brun}_n) = [[I_1, I_2], \cdots, I_{n-1}]_S.$$

Lemma

Lemma 3 *Let I_j be any Lie ideals of a Lie algebra L with $1 \leq j \leq n$. Then*

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Theorem 4

$$L^P(\mathrm{Brun}_n) = [[I_1, I_2], \cdots, I_{n-1}]_S.$$

Proof

It is evident that the symmetric bracket sum $[[I_1, I_2], \cdots, I_{n-1}]_S$ lies in the kernels of all d_i. On the other hand, from lemma 3, $L^P(\mathrm{Brun}_n)$ is given as "fat bracket sum" of $I_1, \cdots, I_{n-1}$ because each element in $\mathcal{K}(n)_1$ is a Lie monomial containing each of $A_{1,n}, \cdots, A_{n-1,n}$. we know that

$$\mathcal{K}(n)_1 \subseteq [[I_1, \cdots, I_{n-1}]] = [[I_1, I_2], \cdots, I_{n-1}]_S.$$

Proposition

Proposition 5 *There is a formula*

$$\mathrm{rank}(L_q(P_n)) = \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{rank}(L_q^P(\mathrm{Brun}_{n-k}))$$

for each n and q.

Let $b_q(P_n) = \mathrm{rank}(L_q(P_n))$ and $b_q^P(\mathrm{Brun}_n) = \mathrm{rank}(L_q^P(\mathrm{Brun}_n))$. we have

$$\begin{pmatrix} b_q(P_n) \\ b_q(P_{n-1}) \\ b_q(P_{n-2}) \\ \vdots \\ b_q(P_1) \end{pmatrix} = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ 0 & 1 & \binom{n-1}{1} & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 1 & \cdots & \binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_q^P(\mathrm{Brun}_n) \\ b_q^P(\mathrm{Brun}_{n-1}) \\ b_q^P(\mathrm{Brun}_{n-2}) \\ \vdots \\ b_q^P(\mathrm{Brun}_1) \end{pmatrix}$$

Let A_n be the coefficient matrix of the above linear equations. Then

$$A_n^{-1} = \begin{pmatrix} 1 & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \cdots & (-1)^{n-1}\binom{n}{n-1} \\ 0 & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \cdots & (-1)^{n-2}\binom{n-1}{n-2} \\ 0 & 0 & 1 & -\binom{n-2}{1} & \cdots & (-1)^{n-3}\binom{n-2}{n-3} \\ 0 & 0 & 0 & 1 & \cdots & (-1)^{n-4}\binom{n-3}{n-4} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Theorem

Theorem 6

$$\mathrm{rank}(L_q^P(\mathrm{Brun}_n)) = \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\mathrm{rank}(L_q(P_{n-k}))$$

for each n and q, where $P_1 = 0$ and, for $m \geq 2$,

$$\mathrm{rank}(L_q(P_m)) = \frac{1}{q}\sum_{k=1}^{m-1}\sum_{d|q}\mu(d)k^{q/d}$$

with μ the Möbis function.

Thanks for your attention!