Representations of the virtual braid groups to the rook algebras

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Definition

Braid group on n strands, denoted by $\mathbb{B}_n$, is group generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1$$
$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \quad \text{if } i = 1, \ldots, n-1$$

Definition

Virtual braid group on n strands, denoted by $V\mathbb{B}_n$, is group with generators:

$$\sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1}$$

and relations:

$$\begin{gathered}
\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \\
\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \quad \text{if } i = 1, \ldots, n-1 \\
\rho_i^2 = e \quad \text{if } i = 1, \ldots, n-1 \\
\rho_{i+1}\rho_i\rho_{i+1} = \rho_i\rho_{i+1}\rho_i \quad \text{if } i = 1, \ldots, n-1 \\
\rho_i\rho_j = \rho_j\rho_i \quad \text{if } |i-j| > 1 \\
\rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1} \quad \text{if } i = 1, \ldots, n-1 \\
\sigma_i\rho_j = \rho_j\sigma_i \quad \text{if } |i-j| > 1
\end{gathered}$$

Definition

Let R_n denote set of n $\times$ n matrices with entries from $\{0, 1\}$ having at most one 1 in each row and in each column.

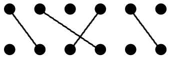
Example for $n = 2$

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Definition

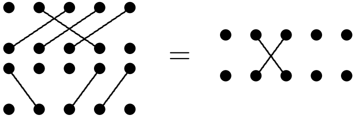
Rook diagram – bipartite graph on two rows of n vertices, one on top of the other forming the boundary of a rectangle, such that each vertex has degree either zero or one.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow$$



Definition

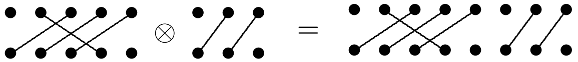
The product, $d_1 d_2$, of two rook diagrams d_1 and d_2 is obtained by stacking d_1 on top of d_2 and deleting any edge from one that connects to a vertex zero degree end from the other.

$d_1 =$  $= d_1 d_2$

$d_2 =$

Definition

Given two diagrams, $a, b \in R_n$, we define the tensor product, denoted $a \otimes b$, to be the result of appending of b to the right of a.



Definition

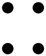
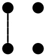
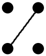
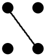
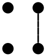
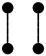
Element of R_n is planar if its diagram can be drawn (keeping inside of the rectangle formed by its vertices) without any edge crossings.

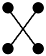
Definition

P_n – planar rook monoid – set of planar diagrams of R_n.

Definition

$\mathbb{C}R_n(\mathbb{C}P_n)$ – (planar) rook algebra – $\mathbb{C}$-algebra generated by $R_n(P_n)$ with multiplication defined using the distributive law and multiplication in $R_n(P_n)$.

$d_1 =$  $d_2 =$  $d_3 =$  $d_4 =$  $d_5 =$  $d_6 =$ 

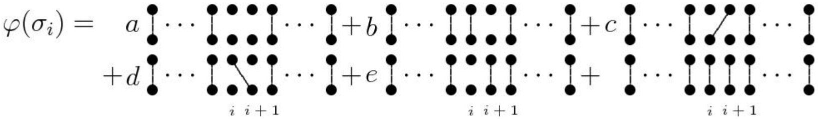
$d_7 =$ 

Definition

$$\varphi(\sigma_i) = a \cdot d_{1i} + b \cdot d_{2i} + c \cdot d_{3i} + d \cdot d_{4i} + e \cdot d_{5i} + f \cdot d_{6i}$$

where I is the identity diagram in P_1 and

$$d_{ji} = I^{\otimes i-1} \otimes d_j \otimes I^{\otimes n-i-1}$$



Theorem

Assuming $a + c + d \neq 1, f = 1$ and $cd \neq 0$, any mapping of the above form is a homomorphism if and only if its coefficients are in one of the following families:

1. $b = e = -1$
2. $a = -c - d,\ b = -1,\ e = -cd$
3. $a = 1 - c - d + cd,\ b = -1,\ e = -cd$

Theorem

There is no reresentation $\psi : V\mathbb{B}_n \to \mathbb{C}P_n$, satisfying the following conditions:

1. $\psi(\rho_i)$ lineral combinations $d_{i,j}$ for $j = 1 \dots 6$
2. Restriction of ψ on $\mathbb{B}_n$ on B_n is φ

Definition

Define mapping $\psi : V\mathbb{B}_n \to \mathbb{C}R_n$ by the next rule:

$$\psi(\sigma_i) = \varphi(\sigma_i)$$
$$\psi(\rho_i) = d_{i,7}$$

Theorem

The mapping ψ, constructed above, is representation of $V\mathbb{B}_n$.

Thank you for attention!