Gröbner-Shirshov bases method in algebra

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1 Introduction

Seminar was organized by the authors in March, 2006. Since then, there were some 30 Master Theses and 4 PhD Theses, about 40 published papers in JA, IJAC, Comm. Algebra, Algebra Coll. and other Journals and Proceedings. There were organized 2 International Conferences (2007, 2009) with E. Zelmanov as Chairman of the Program Committee and several Workshops. We are going to review some of the papers.

Our main topic is Gröbner-Shirshov bases method for different varieties (categories) of linear (Ω-) algebras over a field k or a commutative algebra K over k: associative algebras (including group (semigroup) algebras), Lie algebras, dialgebras, conformal algebras, pre-Lie (Vinberg right (left) symmetric) algebras, Rota-Baxter algebras, metabelian Lie algebras, L-algebras, semiring algebras, category algebras, etc. There are some applications particularly to new proofs of some known theorems.

2 Composition-Diamond lemmas

As it is well known, Gröbner-Shirshov (GS for short) bases method for a class of algebras based on a Composition-Diamond lemma (CD-lemma for short) for the class. A general form of a CD-Lemma over a field k is as follows.

Composition-Diamond lemma Let $M(X)$ be a free algebra of a category M of algebras over k, $(N(X), \leq)$ a linear basis (normal words) of $M(X)$ with an "addmissible" well order and $S \subset M(X)$. TFAE

(i) S is a GS basis (i.e. each "composition" of polynomials from S is "trivial").

(ii) If $f \in Id(S)$, then the maximal word of f has a form $\bar{f} = (a\bar{s}b),\ s \in S,\ a, b \in X^*$.

(iii) $Irr(S) = \{u \in N(X) | u \neq (a\bar{s}b), s \in S, a, b \in X^*\}$ is a linear basis of $M(X|S) = M(X)/Id(S)$.

The main property is $(i) \Rightarrow (ii)$.

CD-lemma for associative algebras

Let $k\langle X\rangle$ be the free associative algebra over a field k generated by X and $(X^*,<)$ a well-ordered free monoid generated by X, $S\subset k\langle X\rangle$ such that every $s\in S$ is monic.

Let us prove $(i)\Rightarrow(iii)$ and define a GS basis.

Let $f=\sum_{i=1}^{n}\alpha_i a_i s_i b_i\in Id(S)$ where each $\alpha_i\in k,\ a_i,b_i\in X^*,\ s_i\in S,\ w_i=a_i\overline{s_i}b_i,\ w_1=w_2=\cdots=w_l>w_{l+1}\geq\cdots$.

For $l=1$, it is ok.

For $l>1,\ w_1=a_1\overline{s_1}b_1=a_2\overline{s_2}b_2$, common multiple of $\overline{s_1},\overline{s_2}$, by definition,

$$w_1 = cwd,\ w = \text{“}lcm\text{”}(\overline{s_1}, \overline{s_2}),\ a_i s_i b_i = w|_{\overline{s_i} \mapsto s_i},\ i = 1, 2,$$

where $lcm(u, v) \in \{ucv, c \in X^* (a\ trivial\ lcm(u, v));\ u = avb,\ a, b \in X^*\ (an\ inclusion\ lcm(u, v));\ ub = av,\ a, b \in X^*,\ |ub| < |u| + |v|\ (an\ intersection\ lcm(u, v)\}$.

Then $a_1 s_1 b_1 - a_2 s_2 b_2 = c(w|_{\overline{s_1} \mapsto s_1} - w|_{\overline{s_2} \mapsto s_2})d = c(s_1, s_2)_w d$. By definition of GS basis, $(s_1, s_2)_w \equiv 0 \ \ mod(S, w)$. So, $a_1 s_1 b_1 - a_2 s_2 b_2 \equiv 0 \ \ mod(S, w_1)$. We can decrease l. By induction, $\bar{f} = a\bar{s}b,\ a, b \in X^*,\ s \in S$.

CD-lemma for Lie algebras over a field

Let $S \subset Lie(X) \subset k\langle X\rangle$ be a nonempty set of monic Lie polynomials, $(X^*, <)$ deg-lex order, $\bar{s}$ means the maximal word of s as non-commutative polynomial,

$$\langle s_1, s_2\rangle_w = [w]_{\overline{s_1}}|_{\overline{s_1}\mapsto s_1} - [w]_{\overline{s_2}}|_{\overline{s_2}\mapsto s_2}, \quad w \in ALSW(X)$$

associative composition with the special Shirshov bracketing.

CD-lemma for Lie algebras over a field. TFAE

(i) S is a Lie GS basis in $Lie(X)$ (any composition is trivial modulo (S, w)).

(ii) $f \in Id_{Lie}(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.

(iii) $Irr(S) = \{[u] \in NLSW(X) \mid u \neq a\bar{s}b,\ s \in S,\ a, b \in X^*\}$ is a linear basis for $Lie(X|S)$.

3 Examples

1. Poincare-Birkhoff-Witt theorem

Let $L = Lie_k(X|S)$ be a Lie algebra over a field k presented by a well-ordered linear basis $X = \{x_i|i \in I\}$ and the multiplication table $S = \{[x_i x_j] - \sum \alpha_{ij}^t x_t | i > j,\ i, j \in I\}$,

$$U(L) = k\langle X|S^{(-)}\rangle,\ \ S^{(-)} = \{x_i x_j - x_j x_i - \sum \alpha_{ij}^t x_t | i > j\}$$

be the universal enveloping associative algebra for L.

Then with deg-lex order on X^*, $S^{(-)}$ is a GS basis and hence following the CD-Lemma for associative algebras a linear basis of $U(L)$ consists of words $x_{i_1} x_{i_2} \ldots x_{i_n}$, $i_1 \leq i_2 \leq \cdots \leq i_n,\ n \geq 0$.

2. Symmetric group S_{n+1}

Symmetric group S_{n+1} is isomorphic to the group

$$\begin{aligned} Coxeter(A_n) &= gp\langle s_1, \ldots, s_n | s_i^2 = 1, \\ &\quad s_{i+1}s_is_{i+1} = s_is_{i+1}s_i, s_is_j = s_js_i, i-j>1\rangle \\ &=: gp\langle \Sigma | S\rangle \end{aligned}$$

with an isomorphism $s_i \mapsto (i, i+1),\ 1 \leq i \leq n$.

A GS basis of $Coxeter(A_n)$ is

$$S \cup \{s_{i+1}s_is_{i-1}\ldots s_js_{i+1} - s_is_{i+1}s_is_{i-1}\ldots s_j | 1 \leq j \leq (i-1)\}.$$

By CD-Lemma for associative algebras a set of normal forms of elements of the group consists of words

$$s_{1j_1}\dots s_{nj_n},\ j_1 \le 2,\dots,j_n \le n+1,$$
$$s_{ij} = s_i s_{i-1}\dots s_j, j \le i, s_{i(i+1)} = 1.$$

Hence $|Coxeter(A_n)| = (n+1)!$ and we are done.

Analogous results are valid for all finite Coxeter groups (of types A_n (before), $B_n, D_n, G_2, F_4, E_6, E_7, E_8$).

3. Lie algebra $sl_{n+1}(k),\ char k \neq 2$

Special linear (trace zero) Lie algebra $sl_{n+1}(k)$ over a field $k,\ char k \neq 2$ is isomorphic to the Lie algebra

$$\begin{aligned} Lie(A_n) \ = \ & Lie(h_i, x_i, y_i, 1 \leq i \leq n | [h_i h_j] = 0, \\ & [x_i y_j] = \delta_{ij} h_i, [h_i x_j] = 2\delta_{ij} x_i, [h_i y_j] = -2\delta_{ij} y_i, \\ & [[x_{i+1}[x_{i+1} x_i]] = 0, [x_j x_i] = 0, \\ & [[y_{i+1}[y_{i+1} y_i]] = 0, [y_j y_i] = 0, j \neq i+1) \end{aligned}$$

with the isomorphism

$$h_i \mapsto e_{ii} - e_{i+1 i+1},\ x_i \mapsto e_{i i+1},\ y_i \mapsto e_{i+1 i},\ 1 \leq i \leq n.$$

A GS basis of $Lie(A_n)$ is the initial relations together with

$$[[x_{i+j}x_{i+j-1}\dots x_{i-1}]x_{i+j-1}],$$
$$[[x_{i+j}\dots x_i][x_{i+j}\dots x_i][x_{i+j}\dots x_{i-1}]],$$
$$j\geq 1,\ i\geq 2,\ i+j\leq n$$

and the same relations for $y_1,\dots,y_n$, where by $[z_1z_2\dots z_m]$ we mean $[z_1[z_2\dots z_m]]$.

By CD-Lemma for Lie algebras a linear basis of $Lie(A_n)$ is

$$h_i,\ [x_ix_{i-1}\dots x_j],[y_iy_{i-1}\dots y_j],\ 1\leq i\leq n,\ j\leq i.$$

Hence $\dim Lie(A_n)=(n+1)^2-1$ and we are done.

Analogous results are valid for all simple Lie algebras (of types A_n (before), $B_n, D_n, G_2, F_4, E_6, E_7, E_8$).

4 PBW theorems

There are 8 PBW theorems that are proved by using GS bases and CD-lemmas.

1. **Lie algebras–associative algebras** (Shirshov)

Let $L = Lie_k(X|S)$, $U(L) = k\langle X|S^{(-)}\rangle$. Then

(i) S is a Lie GS basis $\Leftrightarrow$ S is an associative GS basis.

(ii) In this case, a linear basis of $U(L)$ is

$$u_1u_2\cdots u_t, \ \ u_1 \preceq u_2 \preceq \cdots \preceq u_t \ \ \text{(lex-order)},$$
$$u_i \in Irr(S) \cap ALSW(X).$$

One uses Shirshov factorization theorem:

$$u \in X^*, \ \exists! \ u = u_1 \cdots u_t, \ \ u_1 \preceq \cdots \preceq u_t, \ u_i \in ALSW(X).$$

2. **Lie algebras–pre-Lie algebras** (D. Segal)

$$L = Lie(x_i, i \in I | [x_i, x_j] = \{x_i, x_j\},\ i, j \in I),$$
$$U_{\text{pre-Lie}}(L) = \text{pre-Lie}(X | S^{(-)}),$$

where

$$[x_i, x_j] = \sum \alpha_{ij}^t x_t := \{x_i, x_j\}$$

is the multiplication table of the linear basis $\{x_i | i \in I\}$ of L. Then $L \subset U_{\text{pre-Lie}}(L)$ is a GS basis and $Irr(S)$ is a linear basis of $U_{\text{pre-Lie}}(L)$ by CD-lemma for pre-Lie algebras (Bokut-Chen-Li [19]).

3. **Leibniz algebras–dialgebras** (Aymon, Grivel)

Dialgebra: $a \dashv (b \vdash c) = a \dashv b \dashv c,\ (a \dashv b) \vdash c = a \vdash b \vdash c,\ a \vdash (b \dashv c) = (a \vdash b) \dashv c$ and $\vdash, \dashv$ associative.

Leibniz identity: $[[a,b],c] = [[a,c],b] + [a,[b,c]]$.

Di-commutator: $[a,b] = a \dashv b - b \vdash a$.

$$L = Lei(x_i, i \in I | [x_i, x_j] = \{x_i, x_j\},\ i, j \in I),$$
$$U_{Dialg}(L) = D(X|S^{(-)}).$$

A GS basis is given by Bokut-Chen-Liu [23] and then a linear basis for $U_{Dialg}(L)$ by CD-lemma for dialgebras which implies $L \subset U_{Dialg}(L)$.

4. **Akivis algebras–non-associative algebras** (Shestekov)

Akivis identity: $[[x,y],z]+[[y,z],x]+[[z,x],y]=(x,y,z)+(z,x,y)+(y,z,x)-(x,z,y)-(y,x,z)-(z,y,x)$, where $[x,y]$ is commutator and (x,y,z) is associator.

$$
\begin{aligned}
A = A(x_i, i\in I \mid &[x_i,x_j]=\{x_i,x_j\},\\
&(x_i,x_j,x_t)=\{x_i,x_j,x_t\},\ i,j,t\in I),\\
U(A) = k\{X \mid &S^{(-)}\},\\
S^{(-)} = \{[x_i&,x_j]=\{x_i,x_j\},\\
&(x_i,x_j,x_t)=\{x_i,x_j,x_t\},\ i,j,t\in I\}.
\end{aligned}
$$

A GS basis of $U(A)$ is given by Chen-Li [45] and then $A\subset U(A)$.

5. **Sabinin algebras–modules** (Perez-Izquierdo)

Let $(V, \langle ; \rangle)$ be a Sabinin algebra,

$$\begin{aligned}\widetilde{S}(V) &= T(V)/span\langle xaby - xbay \\ &\qquad + \sum x_{(1)}\langle x_{(2)}; a, b\rangle y | x, y \in T(V), a, b \in V\rangle \\ &\cong mod\langle X|I\rangle_{k\langle X\rangle} \text{ as } k\langle X\rangle\text{-modules}\end{aligned}$$

the universal enveloping module for V, where $I = \{xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle | x \in X^*, a > b,\ a, b \in X\}$.

Then I is a GS basis (Chen-Chen-Zhong [44]) and then $V \subset \widetilde{S}(V)$.

6. **Rota-Baxter algebras–Dendriform algebras** (Chen-Mo [48], Kolesnikov)

Rota-Baxter identity:

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy), \forall x, y \in A.$$

Dendriform identities: $(x \prec y) \prec z = x \prec (y \prec z + y \succ z)$, $(x \succ y) \prec z = x \succ (y \prec z)$, $(x \prec y + x \succ y) \succ z = x \succ (y \succ z)$.

$$\begin{aligned} D &= Den(X | x_i \prec x_j = \{x_i \prec x_j\}, \\ &\qquad x_i \succ x_j = \{x_i \succ x_j\}, x_i, x_j \in X); \\ U(D) &= RB(X | x_i P(x_j) = \{x_i \prec x_j\}, \\ &\qquad P(x_i) x_j = \{x_i \succ x_j\}, x_i, x_j \in X). \end{aligned}$$

Then $D \subset U(D)$.

7. Shirshov's, Cartier's, Cohn's counter examples to PBW for Lie algebras over commutative algebra

Shirshov and Cartier 1958 give counter examples to PBW for Lie algebras over commutative algebra. Cohn posts the conjecture:

$$\begin{aligned}\mathcal{L}_p &= Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1),\\ K &= \mathbf{k}[y_1, y_2, y_3 | y_i^p = 0, i = 1, 2, 3].\end{aligned}$$

$\mathcal{L}_p$ can not be embedded into its universal enveloping associative algebra.

Bokut-Chen-Chen [15] establish GS bases theory for Lie algebras over a commutative algebra. We prove Cohn's conjecture is correct for $p = 2, 3, 5$.

8. **"1/2 PBW theorem"** (Bokut-Fong-Ke [30], Bokut-Chen-Zhang [28])

Conformal Lie algebras–conformal associative algebras; n-conformal Lie algebras–n-conformal associative algebras.

$(C, \langle m\rangle, m \geq 0, D) = C(X|S)$, where X is a linear basis of C and S is the multiplication table.

$w = x_i\langle m\rangle x_j\langle l\rangle x_t,\ i \geq j \geq t.$

If $i > j > t$, the composition is trivial (1/2 PBW). But, if $i = j$ or $j = t$, they may not.

5 Linear bases of free universal algebras

–Bases of free Lie algebras

M. Hall, A.I. Shirshov, Loday, A.G. Kurosh use construction and check axioms.

Hall basis (Bokut-Chen-Li [20]): $Lie(X) = AC(X|S_1)$, S_1 is a anti-commutative GS basis, $Irr(S_1) = Hall(X)$.

Lyndon-Shirshov basis (Bokut-Chen-Li [22]): $Lie(X) = AC(X|S_2)$, S_2 is a anti-commutative GS basis, $Irr(S_2)$ =Lyndon-Shirshov basis in X.

–Loday basis of a free dialgebra

$D(X) = L(X|S)$, L-identity: $(a \vdash b) \dashv c = a \vdash (b \dashv c)$, S a di-GS basis with $Irr(S)$ =Loday basis in X (Bokut-Chen-Huang [18]).

–Bases of a free dentriform algebra

$Den(X) = L(X|S)$, $Irr(S)$=a linear basis of $Den(X)$ (Bokut-Chen-Huang [18]).

–Bases of a free Rota-Baxter algebra

Via GS method for Ω-algebras (Bokut-Chen-Qiu [26]).

–Free inverse semigroup

An associative GS basis is given by (Bokut-Chen-Zhao [29]), $Irr(S)$ is a normal form of free inverse semigroup.

–Free idempotent semigroup (Chen-Yang [52]).

6 Normal forms for groups and semi-groups

–Braid groups

in Artin-Burau generators (Bokut-Chanikov-Shum [10]);

in Artin-Garside generators (Bokut [8]);

in Birman-Ko-Lee generators (Bokut [9]);

in Adyan-Thurston generators (Chen-Zhong [55]).

–Chinese monoid (Chen-Qiu [50])

–Plactic monoid (Bokut-Chen-Chen-Li [16]).

–HNN extension

Britton Lemma and Lyndon-Schupp normal form lemma for an HNN-extension of a group was proved using an associative CD-lemma relative to a "S-partially" monomial order of words (Chen-Zhong [53]).

–one-relator groups

In (Chen-Zhong, [54]), some one-relator groups were studying by means of groups with the standard normal forms (the standard GS bases) in the sense (Bokut, [4, 5]). It is known that any one-relator group can be can be effectively embedded into 2-generator one-relator group $G = gp(x, y | x^{i_1} y^{j_1} \dots x^{i_k} y^{j_k},\ k \geq 1),\ k$ is the depth. It is proved that a group G of depth ≤ 3 is computably embeddable into a Magnus tower of HNN-extensions with the standard normal form of elements. There are quite a lot of examples that support an old conjecture that the result is valid for any depth.

7 Extensions of groups and algebras

In (Chen, [39]), it is dealing with a Schreier extension

$$1 \to A \to C \to B \to 1$$

of a group A by B. C.M. Hall [60] mentioned that for any B it is difficult to find an analogous conditions. Actually this problem was solved in [39] using the GS bases technique. As applications there were given above conditions for cyclic and free abelian cases, as well for the case of HNN-extensions. The same kind of result was established for Schreier extensions of associative algebras (Chen-Zhong [40]).

Chen [40] gives a characterization of algebra extensions by GS method.

8 Embedding algebras

In (Bokut-Chen-Mo [24]), we were dealing with the problem of embedding of countably generated associative and Lie algebras, groups, semigroups, Ω-algebras into (simple) 2-generated ones. We proved some known results (of Higman-Neuman-Neuman, Evance, Malcev, Shirshov) and some new ones using GS bases technique. For example

Theorem 1. Every countable Lie algebra is embeddable into simple 2-generated Lie algebra.

Theorem 2. Every countable differential algebra is embeddable into a simple 2-generated differential algebra.

G. Bergman (Private communication, 2013 [2]) gave an idea how to avoid the restriction on cardinality of the ground field. Now Qiuhui Mo proved that the Bergman's idea works.

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Thank You!