Symmetric cohomology of groups

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Knots, Braids and Automorphism Groups
Novosibirsk
July 2014

- Cohomology of groups is a contravariant functor turning groups and modules over groups into graded abelian groups.
- It came into being with the fundamental work of Eilenberg and MacLane (Ann. Math. 1947).
- The theory was further developed by Hopf, Eckmann, Segal, Serre, and many other mathematicians.
- It has been studied from different perspectives with applications in various areas of mathematics.
- It provides a beautiful link between algebra and topology.

- There are three main equivalent descriptions of cohomology of groups.

$$Algebraic \leftrightarrow Topological \leftrightarrow Combinatorial$$

- Let G be a group and A a G-module.
- For each $n \geq 0$, let $C^n(G, A) = \{\sigma \mid \sigma : G^n \to A\}$ and define $\partial^n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} \partial^n(\sigma)(g_1, \ldots, g_{n+1}) &= g_1\sigma(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n}(-1)^i\sigma(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1}\sigma(g_1, \ldots, g_n). \end{aligned}$$

- Since $\partial^{n+1}\partial^n = 0$, we obtain a cochain complex.
- The nth cohomology of G with coefficients in A is defined as $H^n(G, A) = \operatorname{Ker}(\partial^n)/\operatorname{Im}(\partial^{n-1})$.

Cohomology of groups have concrete group theoretic interpretations.

- $H^0(G, A) = A^G$.
- $H^1(G, A) = Derivations/Principal\ Derivations$.
- Let $\mathcal{E}(G, A) =$ *Set of equivalence classes of extensions of* G *by* A *giving rise to the given action of* G *on* A. Then there is a one-one correspondence between $H^2(G, A)$ and $\mathcal{E}(G, A)$.
- There are also group theoretic interpretations of the functors H^n for $n \geq 3$.

- Let $\Phi : H^2(G, A) \to \mathcal{E}(G, A)$ be the one-one correspondence.
- Under Φ, the trivial element of $H^2(G, A)$ corresponds to the equivalence class of an extension

$$0 \to A \to E \to G \to 1$$

admitting a section $s : G \to E$ which is a group homomorphism.
- An extension $\mathcal{E} : 0 \to A \to E \to G \to 1$ of G by A is called a symmetric extension if there exists a section $s : G \to E$ such that $s(g^{-1}) = s(g)^{-1}$ for all $g \in G$. Such a section is called a symmetric section.
- Let $\mathcal{S}(G, A) = \{[\mathcal{E}] \in \mathcal{E}(G, A) \mid \mathcal{E} \text{ is a symmetric extension}\}$. Then the following question seems natural.

Question 1

What elements of $H^2(G, A)$ corresponds to $\mathcal{S}(G, A)$ under Φ?

- Consider the non-split extension

$$\mathcal{E}_1 : 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \times \mathbb{Z}/2 \xrightarrow{\pi} \mathbb{Z}/4 \to 0,$$

  where $i(n) = (2n, \overline{n})$ and $\pi(n, \overline{m}) = \overline{n + 2m}$.
  Let $s : \mathbb{Z}/4 \to \mathbb{Z} \times \mathbb{Z}/2$ be given by

$$s(\overline{0}) = (0, \overline{0}),\ s(\overline{1}) = (-1, \overline{1}),\ s(\overline{2}) = (0, \overline{1}) \text{ and } s(\overline{3}) = (1, \overline{1}).$$

  Then s is a symmetric section and hence $[\mathcal{E}_1] \in \mathcal{S}(\mathbb{Z}/4, \mathbb{Z})$.
- Consider the split extension

$$\mathcal{E}_2 : 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \times \mathbb{Z}/4 \xrightarrow{\pi} \mathbb{Z}/4 \to 0,$$

  where $i(n) = (n, \overline{0})$ and $\pi(n, \overline{m}) = \overline{m}$. Then $s : \mathbb{Z}/4 \to \mathbb{Z} \times \mathbb{Z}/4$ given by $s(\overline{m}) = (0, \overline{m})$ is a symmetric section and hence $[\mathcal{E}_2] \in \mathcal{S}(\mathbb{Z}/4, \mathbb{Z})$.

- Consider the non-split extensions

$$\mathcal{E}_3 : 0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4 \to 0,$$

where $i(n) = 4n$ and $\pi(n) = \overline{n}$ and

$$\mathcal{E}_4 : 0 \to \mathbb{Z} \xrightarrow{i'} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4 \to 0,$$

where $i'(n) = -4n$ and $\pi'(n) = \overline{n}$.

- These extensions do not admit any symmetric section and hence $[\mathcal{E}_3], [\mathcal{E}_4] \in \mathcal{E}(\mathbb{Z}/4, \mathbb{Z}) - \mathcal{S}(\mathbb{Z}/4, \mathbb{Z})$.
- Thus $\mathcal{S}(G, A) \neq \mathcal{E}(G, A)$ in general.

- Mihai Staic (2009) answered Question 1 for abstract groups.
- Motivated by some problems in constructing invariants of 3-manifolds, Staic introduced a new cohomology theory of groups called symmetric cohomology which classifies symmetric extensions in dimension two.

- For $n \geq 0$, let Σ_{n+1} be the symmetric group on $n+1$ symbols.
- For $1 \leq i \leq n$, let $\tau_i = (i, i+1)$.
- For $\sigma \in C^n(G,A)$ and $(g_1, \ldots, g_n) \in G^n$, define

$$(\tau_1\sigma)(g_1, \ldots, g_n) = -g_1\sigma\big(g_1^{-1}, g_1g_2, g_3, \ldots, g_n\big),$$

$$(\tau_i\sigma)(g_1, \ldots, g_n) = -\sigma\big(g_1, \ldots, g_{i-2}, g_{i-1}g_i, g_i^{-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_n\big)$$

for $1 < i < n$,

$$(\tau_n\sigma)(g_1, \ldots, g_n) = -\sigma\big(g_1, g_2, g_3, \ldots, g_{n-1}g_n, g_n^{-1}\big).$$

- It is easy to see that

$$\tau_i\big(\tau_i(\sigma)\big) = \sigma,$$

$$\tau_i\big(\tau_j(\sigma)\big) = \tau_j\big(\tau_i(\sigma)\big) \text{ for } j \neq i \pm 1,$$

$$\tau_i\Big(\tau_{i+1}\big(\tau_i(\sigma)\big)\Big) = \tau_{i+1}\Big(\tau_i\big(\tau_{i+1}(\sigma)\big)\Big).$$

- Thus there is an action of Σ_{n+1} on $C^n(G,A)$.

- Define $d^j : C^n(G,A) \to C^{n+1}(G,A)$ by
$$d^0(\sigma)(g_1,\dots,g_{n+1}) = g_1\sigma(g_2,\dots,g_{n+1}),$$
$$d^j(\sigma)(g_1,\dots,g_{n+1}) = \sigma(g_1,\dots,g_jg_{j+1},\dots,g_{n+1}) \text{ for } 1 \leq j \leq n,$$
$$d^{n+1}(\sigma)(g_1,\dots,g_{n+1}) = \sigma(g_1,\dots,g_n).$$
Then $\partial^n(\sigma) = \sum_{j=0}^{n+1}(-1)^j d^j(\sigma)$.
- It turns out that
$$\tau_i d^j = d^j\tau_i \text{ if } i < j,$$
$$\tau_i d^j = d^j\tau_{i-1} \text{ if } j+2 \leq i,$$
$$\tau_i d^{i-1} = -d^i,$$
$$\tau_i d^i = -d^{i-1}.$$
- Let $CS^n(G,A) = C^n(G,A)^{\Sigma_{n+1}}$. If $\sigma \in CS^n(G,A)$, then it follows from the above identities that $\partial^n(\sigma) \in CS^{n+1}(G,A)$.
- Thus the action is compatible with coboundary operators.

- We obtain a cochain complex $\{CS^n(G,A), \partial^n\}_{n\geq 0}$. Its cohomology, denoted $HS^n(G,A)$, is called the symmetric cohomology of G with coefficients in A.
- $HS^0(G,A) = A^G = H^0(G,A)$.
- A 1-cochain $\lambda : G \to A$ is symmetric if $\lambda(g) = -g\lambda(g^{-1})$.
  $ZS^1(G,A) =$ group of symmetric derivations.
- A 2-cochain $\sigma : G \times G \to A$ is symmetric if $\sigma(g,h) = -g\sigma(g^{-1}, gh) = \sigma(gh, h^{-1})$.

- The inclusion $CS^*(G, A) \hookrightarrow C^*(G, A)$ induces a homomorphism

$$h^* : HS^*(G, A) \to H^*(G, A).$$

- Clearly h^* is an isomorphism in dimension 0 and is injective in dimension 1. Similar result holds in dimension 2.

Proposition (M. Staic, J. Algebra 2009)

The map $h^* : HS^2(G, A) \to H^2(G, A)$ is injective.

- We now have an answer to Question 1.

Theorem (M. Staic, J. Algebra 2009)

The map $\Phi \circ h^* : HS^2(G, A) \to \mathcal{S}(G, A)$ is a bijection.

- Examples in previous slides corresponds to the fact that $H^2(\mathbb{Z}/4, \mathbb{Z}) = \mathbb{Z}/4$ and $HS^2(\mathbb{Z}/4, \mathbb{Z}) = \mathbb{Z}/2$.

- When the group under consideration is equipped with a topology (or any other structure), it is natural to look for a cohomology theory which also takes the topology (the other structure) into account.
- This lead to various cohomology theories of topological groups.
- Topology was first inserted in the formal definition of cohomology of topological groups in the works of S. -T. Hu (1952), W. T. van Est (1953) and A. Heller (1973).

- Let G be a topological group and A an abelian topological group. We say that A is a topological G-module if there is a continuous action of G on A.
- For each $n \geq 0$, let G^n be the product topological group and $C_c^n(G, A) = \{\sigma \mid \sigma : G^n \to A \text{ is a continuous map}\}$.
- Let $\partial^n : C_c^n(G, A) \to C_c^{n+1}(G, A)$ be the standard coboundary map as used for abstract groups. Then $\{C_c^n(G, A), \partial^n\}_{n \geq 0}$ is a cochain complex.
- The cohomology of this cochain complex, denoted $H_c^*(G, A)$, is called the continuous cohomology of G with coefficients in A.
- Clearly this cohomology coincides with the ordinary cohomology when the groups under consideration are discrete (in particular finite).
- The low dimensional cohomology groups are as expected with $H_c^0(G, A) = A^G$ and $Z_c^1(G, A) =$ group of continuous derivations .

- An extension of topological groups

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

  is an algebraically exact sequence of topological groups with the additional property that i is closed continuous and π is open continuous.
- If we assume that i and π are only continuous, then A viewed as a subgroup of E may not have the relative topology and the isomorphism $E/i(A) \cong G$ may not be a homeomorphism.
- Since i is closed continuous, its an embedding of A onto a closed subgroup of E.
- Two extensions $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ and $0 \to A \xrightarrow{i'} E' \xrightarrow{\pi'} G \to 1$ are called equivalent if there exists an open continuous isomorphism $\phi : E \to E'$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd}$$

- Let G be a topological group and A a topological G-module. Let $\mathcal{E}_c(G, A) =$ *Set of equivalence classes of topological group extensions of G by A giving rise to the given action of G on A.*
- Let $\mathcal{E}_c^0(G, A) = \big\{[\mathcal{E}] \in \mathcal{E}_c(G, A) \mid \mathcal{E}$ admits a global continuous section$\big\}$.
- As in ordinary cohomology, the following result holds for continuous cohomology.

Theorem (Hu, Michigan. Math. J. 1952)

Let G be a topological group and A a topological G-module. Then there is a bijection $\Psi : H_c^2(G, A) \to \mathcal{E}_c^0(G, A)$.

- An extension of topological groups is called topologically split if E is $A \times G$ as a topological space. Clearly an extension admitting a global continuous section is topologically split.
- Extensions admitting a global continuous section are assured by the following result.

Theorem (Shtern, Ann. Global. Ann. Geom. 2001)

Let G be a connected locally compact group. Then any topological group extension of G by a simply connected Lie group admits a global continuous section.

- Let $\mathcal{S}_c^0(G, A) = \{[\mathcal{E}] \in \mathcal{E}_c(G, A) \mid \mathcal{E} \text{ admits a symmetric global cont section}\}$. Then we have the following analogous question.

Question 2

What elements of $H_c^2(G, A)$ corresponds to $\mathcal{S}_c^0(G, A)$ under Ψ?

- Let
$$\mathcal{H}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$
be the 3-dimensional real Heisenberg group. Its a non-abelian topological group (in fact a Lie group).
- Let
$$A = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$
be the center of $\mathcal{H}_3(\mathbb{R})$.
- Since $A \cong (\mathbb{R}, +)$ and $\mathcal{H}_3(\mathbb{R})/A \cong (\mathbb{R}^2, +)$ as topological groups, we have a non-split extension of topological groups
$$\mathcal{E} : 0 \to \mathbb{R} \to \mathcal{H}_3(\mathbb{R}) \to \mathbb{R}^2 \to 0.$$

- Let $s : \mathbb{R}^2 \to \mathcal{H}_3(\mathbb{R})$ be given by

$$s(x,y) = \begin{pmatrix} 1 & x & \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then s is a continuous section. Further s is symmetric

$$s(-x,-y) = \begin{pmatrix} 1 & -x & \frac{xy}{2} \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} = s(x,y)^{-1}.$$

Thus $[\mathcal{E}] \in \mathcal{S}^0_c(G,A)$.

- More examples of topological group extensions admitting a symmetric continuous section are guaranteed by the following result.

Theorem (E. Michael, Ann. Math. 1956)

Let X and Y be real or complex Banach spaces regarded as topological groups with respect to addition. If $\pi : X \to Y$ is a surjective continuous linear map, then there exists a symmetric continuous section $s : Y \to X$.

- Continuity of G-action on A implies that $\tau(\sigma) \in C_c^n(G, A)$ for each $\tau \in \Sigma_{n+1}$ and $\sigma \in C_c^n(G, A)$.
- Let $CS_c^n(G, A) = C_c^n(G, A)^{\Sigma_{n+1}}$. Then $\{CS_c^n(G, A), \partial^n\}_{n \geq 0}$ is a cochain complex. Its cohomology, denoted $HS_c^n(G, A)$, is called the symmetric continuous cohomology of G with coefficients in A.
- $HS_c^0(G, A) = A^G = H_c^0(G, A)$.
- $ZS_c^1(G, A) =$ group of symmetric continuous derivations.

- The inclusion $CS^*_c(G, A) \hookrightarrow C^*_c(G, A)$ induces a homomorphism

$$h^*_c : HS^*_c(G, A) \to H^*_c(G, A).$$

- h^*_c is an isomorphism in dimension 0 and is injective in dimension 1.

Proposition (S, HHA 2013)

The map $h^*_c : HS^2_c(G, A) \to H^2_c(G, A)$ is injective.

- Following theorem answers Question 2.

Theorem (S, HHA 2013)

The map $\Psi \circ h^*_c : HS^2_c(G, A) \to \mathcal{S}^0_c(G, A)$ is a bijection.

- Let G be a topological group and A a topological G-module. Restriction to the underlying abstract group structure gives the homomorphisms

$$r^* : H^2_c(G,A) \to H^2(G,A) \text{ and}$$

$$r^*_s : HS^2_c(G,A) \to HS^2(G,A)$$

making the following diagram commute.

$$\begin{tikzcd} HS^2_c(G,A) \arrow[r, "r^*_s"] \arrow[d, "h^*_c"] & HS^2(G,A) \arrow[d, "h^*"] \\ H^2_c(G,A) \arrow[r, "r^*"] & H^2(G,A) \end{tikzcd}$$

- We have shown that the vertical maps h^*_c and h^* are injective.

- Using results of C. C. Moore (Trans. AMS 1968, IHES 1976) on measurable cohomology, we obtain the following result.

Proposition

Let G be a perfect group satisfying either of the following conditions:

1. G is a profinite group and A a discrete G-module.
2. G is a Lie group and A a finite dimensional G-vector space.

Then the restriction map $r_s^* : HS_c^2(G, A) \to HS^2(G, A)$ is injective.

- The map r_s^* is not injective in general.
- Let X be an infinite dimensional complex Banach space and A be a non-complemented subspace of X. Then the quotient map $\pi : X \to X/A$ admits a symmetric continuous section by the result of E. Michael. Since A is non-complemented in X, the extension

$$\mathcal{E} : 0 \to A \to X \to X/A \to 0$$

is non-split as an extension of topological groups and hence $[\mathcal{E}] \neq 0$ in $HS_c^2(X/A, A)$.

But X is isomorphic to $A \times X/A$ as an abelian group and hence $r_s^*\big([\mathcal{E}]\big) = 0$ in $HS^2(X/A, A)$.

- The map r_s^* is not surjective in general.
- Consider the extension of abstract groups

$$\mathcal{E} : 0 \to \mathbb{R} \to \mathcal{H}_3(\mathbb{R}) \to \mathbb{R}^2 \to 0.$$

Then $[\mathcal{E}] \neq 0$ in $HS^2(\mathbb{R}^2, \mathbb{R})$.

Consider $(\mathbb{R}, +)$ as a topological group with discrete topology and $(\mathbb{R}^2, +)$ as a topological group with usual topology. Then there is no topology on $\mathcal{H}_3(\mathbb{R})$ making $\mathcal{E}$ into an extension of topological groups inducing the underlying abstract group extension.

Hence $[\mathcal{E}] \in HS^2(\mathbb{R}^2, \mathbb{R})$ has no preimage in $HS^2_c(\mathbb{R}^2, \mathbb{R})$ under r_s^*.

- A profinite group is a topological group which is isomorphic to the inverse limit of an inverse system of discrete finite groups.
- Profinite groups form a special class of topological groups.
- The symmetric continuous cohomology of profinite groups behave well with the symmetric cohomology of finite groups.

Theorem (S, HHA 2013)

The symmetric continuous cohomology of a profinite group with coefficients in a discrete module equals the direct limit of the symmetric cohomology of finite groups.

- Let G be a Lie group and A a smooth G-module.
- For each $n \geq 0$, let $C_s^n(G, A) = \{\sigma \mid \sigma : G^n \to A$ is a smooth map$\}$.
- Let $\partial^n : C_s^n(G, A) \to C_s^{n+1}(G, A)$ be the standard coboundary map. Then $\{C_s^n(G, A), \partial^n\}_{n \geq 0}$ is a cochain complex giving the smooth cohomology of G with coefficients in A. We denote it by $H_s^*(G, A)$.
- $H_s^0(G, A) = A^G$.

  $Z_s^1(G, A) =$ group of smooth derivations.

- Smoothness of G-action on A implies that $\tau(\sigma) \in C_s^n(G, A)$ for each $\tau \in \Sigma_{n+1}$ and $\sigma \in C_s^n(G, A)$.
- Let $CS_s^n(G, A) = C_s^n(G, A)^{\Sigma_{n+1}}$. Then $\{CS_s^n(G, A), \partial^n\}_{n \geq 0}$ is a cochain complex giving the symmetric smooth cohomology of G with coefficients in A. We denote it by $HS_s^n(G, A)$.
- $HS_s^0(G, A) = A^G$.
- $ZS_s^1(G, A) =$ group of symmetric smooth derivations.

- An extension of Lie groups

$$0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

  is an algebraic short exact sequence of Lie groups with the additional property that both i and π are smooth homomorphisms and π admits a local smooth section $s : U \to E$, where $U \subset G$ is an open neighbourhood of identity.

- The existence of a local smooth section means that E is a principal A-bundle over G with respect to the left action of A on E given by $(a, e) \mapsto i(a)e$ for $a \in A$ and $e \in E$.
- Since an extension of Lie groups is a principal bundle, it is a trivial bundle (E is $A \times G$ as a smooth manifold) if and only if it admits a global smooth section.

- Two extensions of Lie groups $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ and $0 \to A \xrightarrow{i'} E' \xrightarrow{\pi'} G \to 1$ are said to be equivalent if there exists a smooth isomorphism $\phi : E \to E'$ with smooth inverse such that the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & G \arrow[r] & 1.
\end{tikzcd}$$

- Let G be a Lie group and A a smooth G-module. Let $\mathcal{S}_s(G, A) =$ *Set of equivalence classes of Lie group extensions of G by A admitting a symmetric global smooth section and giving rise to the given action of G on A.*

Theorem (S, HHA 2013)

Let G be a Lie group and A a smooth G-module. Then there is a one-one correspondence $HS^2_s(G, A) \leftrightarrow \mathcal{S}_s(G, A)$.

- An extension of Lie groups $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ can be thought of as an extension of topological groups by considering only the underlying topological group structure.
- This gives the restriction homomorphism

$$r_s^* : HS_s^2(G, A) \to HS_c^2(G, A).$$

- Hilbert's fifth problem: Is every locally Euclidean topological group necessarily a Lie group?
- The problem has a positive solution due to Gleason and Montgomery (Ann. Math. 1952).
- Contrary to the topological case, using solution of Hilbert's fifth problem, we have the following result.

Theorem (S, HHA 2013)

Let G be a Lie group and A a smooth G-module. Then the restriction map $r_s^* : HS_s^2(G, A) \to HS_c^2(G, A)$ is an isomorphism.

- The following questions seems natural.

1. Does there exists a Lyndon-Hochschild-Serre type spectral sequence for symmetric cohomology of groups?
2. Is it possible to define a symmetric cohomology of Lie algebras? How does this relate to the symmetric cohomology of Lie groups?

- It seems possible to define a symmetric cohomology of Lie algebras. M. Staic suspect that it is equal to the usual cohomology.

- Let G be a group and $\mathbb{C}^\times$ a trivial G-module. Then the Schur multiplier of G is defined as $\mathcal{M}(G) := H^2(G, \mathbb{C}^\times)$.
- It turns out that the Schur multiplier $\mathcal{M}(G)$ of a finite group G is a finite abelian group. Finding bounds on the order of $\mathcal{M}(G)$ is an active area of research and has wide range of applications.
- We define symmetric Schur multiplier of G as $\mathcal{M}S(G) := HS^2(G, \mathbb{C}^\times)$.
- Symmetric Schur multiplier $\mathcal{M}S(G)$ of a finite group G is a finite group.The following questions seems interesting.

1. Find bounds on the order of $\mathcal{M}S(G)$.
2. Find bounds on the order of $\mathcal{M}(G)/\mathcal{M}S(G)$.
3. Relate $\mathcal{M}S(G)$ to other interesting subgroups of $\mathcal{M}(G)$.

Thank You All