RESIDUAL PROPERTIES AND LINEAR REPRESENTATIONS OF GROUPS

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Definition 1

A group G is said to be *linear* if it is isomorphic to a subgroup of the general linear group $\mathrm{GL}_n(F)$ for some n and a field F. This isomorphism is called the *linear (or matrix) representation* of the group G and the integer n is called the *degree* of the representation.

Definition 2

Let $\mathfrak{X}$ be a class of groups. A group G is referred to as *super-residually* $\mathfrak{X}$ if for every finite subset $X \subset G$ there exists normal subgroup $N \lhd G$ such that $G/N \in \mathfrak{X}$ and $xN \neq yN$ for all $x, y \in X$. A group G is referred to as *residually* $\mathfrak{X}$ if for every element $x \in G$ there exists normal subgroup $N \lhd G$ such that $G/N \in \mathfrak{X}$ and $xN \neq N$.

History of the problem

Mal'cev [1] and Nisnevich [2] independently found a connection between the linear representation of a finitely generated group and its super-residuality $\mathcal{L}(n, F)$ where $\mathcal{L}(n, F)$ is a class of $n \times n$-matrix groups with coefficients from fields.

[1] *Mal'cev A. I.* On the faitful representation of ifinite groups by matrices // [in Russian], Math. Sb. Vol. 8, No. 3, 405-423 (1940); English transl.: Am. Math. Soc. Transl. Ser., Vol. 2, No. 45, 1-18 (1965).

[2] *Nisnevich V. L.* On groups isomorphically representable by matrices over commutative field // [in Russian], Math. Sb. Vol. 8, No. 3, 395-403 (1940).

Lubotzky [3] gave a criterion for finitely generated groups to be linear with complex coefficients. The criterion means that these groups contain a finite index subgroup that is super-residually finite p-group of bounded finite rank.

[3] *Lubotzky A.* A group theoretic characterzation of linear groups // J. Algebra, Vol. 113, 207-214 (1988).

History of the problem

In the paper [4] some sufficient conditions for an isomorphic representation over a field of a group by matrices and a criterion of the linear representation for finitely generated groups are presented. The criterion is based on the fact that a group must be super-residually $\mathcal{L}(n, R)$ where $\mathcal{L}(n, R)$ is a class of $n \times n$-matrix groups with coefficients from some class of associative commutative rings involving all fields. This result generalizes a similar criterion due to Mal'cev [1].

[4] *Bryukhanov O. V.* Approximation properties and linearty of groups // J. of Math. Sciences, Vol. 188, No. 4, 354-358 (2013).

Definition 3

A group G is referred to as *uniformly super-residually* $\mathcal{L}(n, R)$ (*uniformly super-residually matrix group of degree* n) if there is a sequence of homomorphisms $\varphi_i : G \to \mathrm{GL_n}(\mathfrak{o}_i)$, where $\mathfrak{o}_i$ are associative commutative rings with identity. Moreover, in each $\mathfrak{o}_i$ there is a descending sequence of ideals

$$\mathfrak{o}_i \rhd P_{i1} \rhd P_{i2} \rhd \cdots \rhd P_{ij} \rhd \cdots$$

such that for every nonidentity element $g \in G$ and any pair of numbers $j_1, j_2 \in \mathbb{N}$ there is a triple $m, l, k \in \mathbb{N}$ such that for $i > l$:
a) $\varphi_i(g) - E \notin \mathbf{Mat_n}(P_{im})$;
b) in the rings $\mathfrak{o}_i$, from $x \notin P_{ij_1}$ and $y \notin P_{ij_2}$ it follows that $xy \notin P_{ik}$.

Results

Theorem A

A uniformly super-residually $\mathcal{L}(n, R)$ group G is isomorphically represented by matrices with degree n over a field F.

Theorem B

A finitely generated group G is linear if and only if it is a uniformly super-residually $\mathcal{L}(n, R)$.

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Simple examples of described rings R are factor-rings of the rings such as ring of integer numbers $\mathbb{Z}$, rings of polynomials $\mathrm{F}_p[x_1, \ldots, x_s]$ and rings of integer p-adic numbers $\mathbb{Z}_{p^\infty}$.

Residual nilpotentness of fundamental groups of 3-dimensional compact Sol-manifolds

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Preliminary

One of the eight possible 3-dimensional model geometries that were described in Thurston's book [1] is Sol-geometry. The space of 3-dimensional model Sol-geometries is $\mathbb{R}^3$ and sol-metric is

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2.$$

The group isometries of 3-dimensional Sol-geometry is a subgroup of the diffeomorphisms group of $\mathbb{R}^3$.

[1] *Thurston W. P.* Thre-Dimensional Geometry and Topology, V 1. Princeton, NJ: Princeton University Press, 1997.

By Thurston ([1], p. 190), its connected component of identity G_0 consist of following transformations of $\mathbb{R}^3$

$$(x, y, t) \to (e^{t_0}x + x_0,\ e^{-t_0}y + y_0,\ t + t_0).$$

The full group of sol-isometries G is a result of adding three transformations:

$$(x, y, t) \to (-x, y, t),\ (x, y, t) \to (x, -y, t),\ (x, y, t) \to (y, x, -t).$$

The last three transformations generate the dihedral group $D(4)$ of eight elements. So

$$G_0 = \mathbb{R}^2 \rtimes \mathbb{R},\ G = G_0 \rtimes \mathrm{D}(4),$$

the index of G_0 is equal 8.

Given $H = G_0 \cap \Gamma$, $H \simeq \mathbb{Z}^2 \leftthreetimes_M \mathbb{Z}$ and is a subgroup of integer matrixes from $\mathrm{SL}_3(\mathbb{Z})$:

$$H = \langle \begin{pmatrix} 1 & v \\ 0 & M^i \end{pmatrix} \mid \ i \in \mathbb{Z}, v \in \mathbb{Z}^2 \rangle,$$

where $M \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr} M > 2$ ([1], Theorem 4.7.13).
There Γ is a discrete cocompact subgroups of the full group of model 3-dimensional Sol-geometry isometries G.

Results

A matrixes representation of the full group of 3-dimensional model Sol-geometry isometries is described in the following proposition. All matrices of the representation are degree 3 with real coefficients.

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Theorem A

The full group of isometries of model 3-dimensional Sol-geometry is isomorphic to the group generated by the matrix set

$$\begin{pmatrix} 1 & x & y \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},\ x, y, t \in \mathbb{R},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Results

For any discrete cocompact subgroup Γ of full group of isometries of model 3-dimensional Sol-geometry that is represented as above there exists matrix $T \in \mathrm{GL}_3(\mathbb{R})$ that $T^{-1}\Gamma T < \mathrm{GL}_3(\mathbb{Z})$. This fact and some more are proven in the next proposition.

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Theorem B

Given Γ a discrete cocompact subgroups of the full group of model 3-dimensional Sol-geometry isometries G, Γ can be represented by integer matrixes from $\mathrm{GL}_3(\mathbb{Z})$ herewith

$$\Gamma \cap G_0 = \langle \begin{pmatrix} 1 & (2-\mathrm{tr}M)v \\ 0 & M^i \end{pmatrix} \mid \; i \in \mathbb{Z}, v \in \mathbb{Z}^2 \rangle$$

where M is fixed by Γ and is a matrix from $\mathrm{SL}_3(\mathbb{Z})$ with $\mathrm{tr}M > 2$.

Results

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It means the same that the intersection $\bigcap\limits_{r=1}^{\infty} \gamma_r(G)$ is trivial where $\gamma_r(G) = \langle [g_1, \ldots, g_r] \mid g_i \in G \rangle$ is the r^{th} central of G group, commutators $[g_1, \ldots, g_r]$ are defined as

$$[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2, \quad [g_1, \ldots, g_r, g_{r+1}] = [[g_1, \ldots, g_r], g_{r+1}].$$

Results

The matrix representation of subgroup Γ allows to give the criterion of residual nilpotentness for this subgroup Γ.

Theorem C

Let Γ be a discrete cocompact subgroup of isometries 3-dimensional Sol-geometry and $H = \Gamma \cap G_0 = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ where $M \in \mathrm{SL}_2(\mathbb{Z}), \mathrm{tr} M > 2$, then:

(a) given $\Gamma = H$, Γ is residually nilpotent if and only if $\mathrm{tr} M > 3$;

(b) given $\Gamma > H$, Γ is residually nilpotent if and only if:

- $\mathrm{tr} M \equiv 1 (\mathrm{mod} 2), \mathrm{tr} M > 3$ and Γ acts without torsions on a line bundle of the model 3-dimensional Sol-geometry space;
- $\mathrm{tr} M \equiv 0 (\mathrm{mod} 2)$. Moreover, it is residually finite 2-group.

Linear representations of some extensions (group theory constructions)

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The following question is well known. When do any group theory constructions transform linear groups to a linear group?
There are various methods to verify the linearity of abstract groups. And the Merzljakov method of splittable coordinates is a very helpful one.

In the paper [1] the matrix representability of nilpotent products $A(n)B$, $n \geq 2$, for linear groups A, B were studied, here

$$A(n)B = A * B/[A, B] \cap \gamma_{n+1}(A * B).$$

[1] *Bryukhanov O. V.* Matrix representation and structure of groups // [in Russian] Algebra and Model Theory 4., Collection of papers. Edited by A.G. Pinus and K.N. Ponomarev. Novosibirsk State Technical University, 15-29 (2003).

More precisely, the criterion of the matrix representability of nilpotent products $A(n)B, n \geq 2$, for finitely generated linear groups A, B was presented.

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In particular it implies that if $F(X), F(Y)$ are free groups then nilpotent products $F(X)(n)F(Y), n \geq 2$ are matrix representable over fields of null characteristic and are not represented by matrix over field of prime characteristic.

Results

In the paper [2] the authors found some sufficient conditions for the semi-direct product of two linear groups to be linear by using the Merzljakov's method . As consequence there were proved the linearity of the groups such as some
$\mathrm{Hol} B_n, n \geq 2,$
$\mathrm{Hol} F_2,$
$F_\varphi(X) = \langle X, t \parallel t^{-1}xt = \varphi(x), x \in X \rangle,$
where φ is virtually inner automorphism of F_2. In all cases faithful linear representations were constructed in the explicit form.

[2] *Bardakov V. G., Bryukhanov O. V.* On linear representation of some extensions // [in Russian] Vestnik, Quart. J. of Novosibirsk State Univ., Serias: Math., mech. and informatics., Vol. 7, No. 3, 45-58 (2007).

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