Orlicz spaces and first cohomology of discrete groups

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Introduction

Inspired by works of Puls and Martin–Valette (see [1], [2] and [3]) on first L^p-cohomology of discrete groups and p-harmonic functions, we introduce by analogy the notion of the discrete Φ-Laplacian and prove a decomposition theorem for the space of Φ-Dirichlet functions, where Φ is an N-function belonging to the class $\Delta_2(0) \cap \nabla_2(0)$. According to the idea, we study the nonreduced and reduced first cohomology of a (finitely generated) discrete group G with coefficients in the left regular representation of G in the Orlicz space $\ell^\Phi(G)$ and show that if G contains an infinite normal amenable subgroup with infinite centralizer then the cohomology space $H^1(G, \ell^\Phi(G)) = 0$. We also prove a theorem about the triviality of the first cohomology space for a wreath product of two groups the first of which is nonamenable.

- M. Bourdon, F. Martin, and A. Valette, Vanishing and non-vanishing for the first L^p-cohomology of groups, *Comm. Math. Helv.*, **80** (2005), no. 2, 377–389.
- F. Martin and A. Valette, On the first L^p-cohomology of discrete groups, *Groups Geom. Dyn.* **1** (2007), no. 1, 81–100.
- M. Puls, The first L^p-cohomology of some finitely generated groups and p-harmonic functions, *J. Funct. Anal.* **237** (2006), no. 2, 391–40.

Definition

A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an *N-function* if it can be represented as

$$\Phi(x) = \int_0^{|x|} \varphi(t)\, dt,$$

where the function $\varphi(t)$ is defined for $t \geqslant 0$, non-decreasing, right-continuous, $\varphi(t) > 0$ if $t > 0$, $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(t) = \infty$. In what follows, Φ' stands for this function φ.
An N-function Φ has the following properties:

- $\Phi(x) > 0$, if $x > 0$;
- Φ is even and convex;
- $\lim\limits_{x\to 0} \dfrac{\Phi(x)}{x} = 0$, $\lim\limits_{x\to\infty} \dfrac{\Phi(x)}{x} = +\infty$.

Orlicz Spaces

Definition

If Φ is an N-function then the function given by

$$\Psi(x) = \int_0^x (\Phi')^{-1}(t)\, dt, \quad \text{where } (\Phi')^{-1}(x) = \sup_{\Phi'(t) \leqslant x} t,$$

is called *complementary to* Φ.

Definition

An function Φ is called *uniformly convex* if, given any $a \in (0,1)$, there exists $\beta(a) \in (0,1)$ such that

$$\Phi\left(\frac{u + bu}{2}\right) \leq \frac{1}{2}(1 - \beta(a))(\Phi(u) - \Phi(bu))$$

for any $b \in [0,1]$ and $u \geqslant 0$.

Definition

An N-function Φ is said to satisfy the Δ_2-*condition for small* x, which is written as $\Phi \in \Delta_2(0)$, if there exist constants $x_0 > 0$, $K > 2$ such that $\Phi(2x) \leq K\Phi(x)$ for $0 \leq x \leq x_0$; and it satisfies the ∇_2-*condition for small* x, which is denoted symbolically as $\Phi \in \nabla_2(0)$ if there are constants $x_0 > 0$ and $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for $0 \leq x \leq x_0$.

Definition

The Orlicz class $\tilde{\ell}^\Phi(X)$ is the set of real-valued functions on X for which

$$\rho_\Phi(x) := \sum_{x \in X} \Phi(f(x)) < \infty.$$

Orlicz Spaces

We will use the notation

$$\tilde{\ell}_1^{\Phi}(X) = \left\{ f \in \tilde{\ell}^{\Phi}(X) \;\middle|\; \sum_{x \in X} \Phi(f(x)) \leq 1 \right\}$$

Definition

The linear space

$$\ell^{\Phi}(X) = \{f : X \to \mathbb{R} \, : \, \rho_{\Phi}(af) < \infty \text{ for } \textit{some } a > 0\}$$

is called an *Orlicz space* on X.

Remark. As is well known, $\tilde{\ell}^{\Phi}(X)$ is a linear space if and only if $\Phi \in \Delta_2(0)$.

Definition

If $f \in \ell^{\Phi}(X)$ then the *Orlicz norm* of f is, by definition,

$$\|f\|_{\Phi} := \|f\|_{\ell^{\Phi}(X)} := \sup_{u \in \tilde{\ell}_1^{\Psi}} \left| \sum_{x \in X} f(x)u(x) \right| .$$

Definition

The *gauge* (or *Luxemburg*) *norm* of a function $f \in \ell^{\Phi}(X)$ is defined by the formula

$$\|f\|_{(\Phi)} := \|f\|_{\ell^{(\Phi)}(X)} := \inf\left\{ k > 0 \, : \, \rho_{\Phi}\left(\frac{f}{k}\right) \leq 1 \right\}.$$

It is well known that the Orlicz and gauge norms are equivalent, namely:

$$\|f\|_{(\Phi)} \leq \|f\|_{\Phi} \leq 2\|f\|_{(\Phi)}.$$

Let G be a topological group and let V be a topological G-module, i.e., a real or complex topological vector space endowed with a linear representation $\pi : G \times V \to V$, $(g, v) \mapsto \pi(g)v$. The space V is called a *Banach G-module* if V is a Banach space and π is a representation of G by isometries of V.

Introduce the notation:

$Z^1(G, V) := \{b : G \to V \text{ continuous } \mid b(gh) = b(g) + \pi(g)b(h)\}$;
$B^1(G, V) = \{b \in Z^1(G, V) \mid (\exists v \in V)\,(\forall g \in G)\ b(g) = \pi(g)v - v\}$;
$H^1(G, V) = Z^1(G, V)/B^1(G, V)$ (*1-cohomology with coefficients in V*)

Endow $Z^1(G, V)$ with the topology of uniform convergence on compact subsets of G and denote by $\overline{B}^1(G, V)$ the closure of $B^1(G, V)$ in this topology. The quotient $\overline{H}^1(G, V) = Z^1(G, V)/\overline{B}^1(G, V)$ is called the *reduced 1-cohomology* of G with coefficients in the G-module V.

Φ-Harmonic Functions

Let G be a finitely generated group with finite generating set S, and suppose that G acts on a countable set X.
If A is an abelian group then denote by A^X the abelian group of all functions $f : X \to A$. Denote by $\lambda_X : G \to A^X$ the *permutation representation* of G on A^X:

$$\lambda_X(g)f(x) = f(g^{-1}x), \quad f \in A^X, \ g \in G.$$

This turns A^X into a G-module.

Introduce the space of Φ-*Dirichlet-finite functions*

$$\begin{aligned}\mathcal{D}^{\Phi}(X) &= \{f \in \mathcal{F}(X) \mid \|\lambda_X(g)f - f\|_{\ell^{\Phi}(X)} < \infty \text{ for all } g \in G\} \\ &= \{f \in \mathcal{F}(X) \mid \|\lambda_X(s)f - f\|_{\ell^{\Phi}(X)} < \infty \text{ for all } s \in S\}.\end{aligned}$$

Let $\mathcal{D}^{\Phi}(X)^G$ be the space of functions that are constant on G-orbits of X. Endow $\mathscr{D}^{\Phi}(X) = \mathcal{D}^{\Phi}(X)/\mathcal{D}^{\Phi}(X)^G$ with the norm

$$\|f\|_{\mathcal{D}^{\Phi}(X)} = \sum_{s \in S} \|\lambda_X(s)f - f\|_{\ell(\Phi)}$$

Φ-Harmonic Functions

Define a linear map $\alpha : \mathcal{D}^{\Phi}(X) \to Z^1(G,\ \ell^{\Phi}(X))$ by setting $\alpha(f)(\gamma) = \lambda_X(\gamma)f - f$. The map induced by α on $\mathscr{D}^{\Phi}(X)$ is an injection since $\mathcal{D}^{\Phi}(X)^G$ is the kernel of α.

Theorem 1

Suppose that a finitely generated group G acts freely on a countable set X. Then $\alpha : \mathscr{D}^{\Phi}(X) \to Z^1(G, \ell^{\Phi}(X))$ is a topological isomorphism, which implies the following:

1. $H^1(G, \ell^{\Phi}(X)) \cong \mathscr{D}^{\Phi}(X)/\ell^{\Phi}(X)$
2. $\overline{H}^1(G, \ell^{\Phi}(X)) \cong \mathscr{D}^{\Phi}(X)/\overline{\ell^{\Phi}(X)}$

Φ-Harmonic Functions

Definition

Suppose that G is a finitely generated group, S is its finite generating set, and G acts on a countable set X. Define $\Delta_\Phi : \mathcal{F}(X) \to \mathcal{F}(X)$ by

$$(\Delta_\Phi f)(x) := \sum_{s \in S} \Phi'(f(s^{-1}x) - f(x)) \text{ for } f \in \mathcal{F}(X) \text{ and } x \in X$$

A function $f \in \mathcal{D}^\Phi(X)$ is said to be Φ-*harmonic* if $(\Delta_\Phi f)(x) = 0$ for every $x \in X$. Denote the set of Φ-harmonic functions on X by $\mathcal{HD}^\Phi(X)$.

Introduce the pairing $\langle \Delta_\Phi *, * \rangle : \mathcal{D}^\Phi(X) \times \mathcal{D}^\Phi(X) \to \mathbb{R}$ as

$$\langle \Delta_\Phi h, f \rangle := \sum_{x \in X} \sum_{s \in S} \Phi'(h(s^{-1}x) - h(x))(f(s^{-1}x) - f(x))$$

Φ-Harmonic Functions

Recall that if V is a real vector space then a functional $\rho : V \to [0, \infty]$ is called a *modular* on V if the following hold for any $x, y \in V$:

1. $\rho(0) = 0$;
2. $\rho(-x) = \rho(x)$;
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$;
4. $\rho(x) = 0$ implies $x = 0$.

Let the space $\mathcal{D}^\Phi(X)$ be endowed with the modular $\rho : \mathcal{D}^\Phi(X) \to \mathbb{R}^+$,

$$\rho(f) = \sum_{s \in S} \sum_{x \in X} \Phi\big(f(s^{-1}x) - f(x)\big).$$

Φ-Harmonic Functions

The *Gâteaux differential* of ρ at a point $f \in \mathcal{D}^{\Phi}(X)$ is defined as

$$\rho'_f(g) = \lim_{t \to 0^+} \frac{\rho(f + tg) - \rho(f)}{t}.$$

It is easy to check that $\rho'_f(g) = \langle \Delta_{\Phi} f, g \rangle$.

Φ-Harmonic Functions

Proposition 1

Assume that Φ is a continuously differentiable strictly convex N-function. Let $f_1, f_2 \in \mathcal{D}^{\Phi}(X)$. Then $f_1 - f_2 \in \mathcal{D}^{\Phi}(X)^G$ if and only if $\langle \Delta_\Phi f_1, f_1 - f_2 \rangle = \langle \Delta_\Phi f_2, f_1 - f_2 \rangle$

For every $x \in X$, define a function $\delta_x : X \to \mathbb{R}$ by

$$\delta_x(t) = \begin{cases} 1, & \text{if } t = x \\ 0, & \text{if } t \neq x \end{cases}$$

Lemma 1

The following are equivalent for $f \in \mathcal{D}^{\Phi}(X)$:

1. $f \in \mathcal{HD}^{\Phi}(X)$;
2. $\langle \Delta_\Phi f, \delta_x \rangle = 0$ *for all* $x \in X$;
3. $\langle \Delta_\Phi f, h \rangle = 0$ *for all* $h \in (\overline{\ell^{\Phi}(X)})_{\mathcal{D}^{\Phi}}$.

Theorem 2

Suppose that $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ is a continuously differentiable strictly convex function. Let G be a finitely generated group acting on a countable set X. Then for $f \in \mathcal{D}^\Phi(X)$ there exists a decomposition $f = u + h$, where $u \in (\overline{\ell^\Phi(X)})_{\mathcal{D}^\Phi}$ and $h \in \mathcal{HD}^\Phi(X)$. It is unique up to an element of $\mathcal{D}^\Phi(X)^G$.

Proof/Step 1

Since $\Phi \in \Delta_2(0) \cap \nabla_2(0)$, the space $\ell^\Phi(X)$ is reflexive.
Let $d = \inf_{g \in (\overline{\ell^\Phi(X)})_{\mathcal{D}^\Phi}} \rho(f - g)$.
Consider the set $B = \{g \in (\overline{\ell^\Phi(X)})_{\mathcal{D}^\Phi} \mid \rho(f - g) \leq d + 1\}$.
It is not hard to check that B is a bounded closed convex set in the reflexive Banach space (Actually, we need Δ_2-regularity).

Φ-Harmonic Functions

Proof/Step 1

Hence, appealing to Kakutani's Theorem, we can conclude that B is compact in the weak topology.

Proof/Step 2

Consider the weakly lower semi-continuous functional

$$F(g) = \rho(f - g),\ g \in (\overline{\ell^{\Phi}(X)})_{\mathcal{D}^{\Phi}}$$

In view of **Step 1**, the functional attains its minimum d on B. Let $F(u) = d$ and $h = f - u$. For $v \in \ell^{\Phi}(X)$, consider the smooth function

$$F_v(t) = \rho(f - (u - tv)), \quad t \in \mathbb{R}.$$

Φ-Harmonic Functions

Proof/Step 2

Obviously, the minimum of F is attained at $t = 0$, which means that

$$\left.\frac{dF_v(t)}{dt}\right|_{t=0} = \rho_h'(v) = \langle \Delta_\Phi h,\, v\rangle = 0, \text{ for all } v \in \ell^\Phi(X).$$

Therefore, $\langle \Delta_\Phi h,\, \delta_x\rangle = 0$ for all $x \in X$, and, consequently, $h \in \mathcal{HD}^\Phi(X)$ by **Lemma 1**.

Proof/Step 3

Prove the uniqueness. Suppose that $f = u_1 + h_1 = u_2 + h_2$.
Appealing to **Lemma 1**, we have
$\langle \Delta_\Phi h_1,\, h_1 - h_2\rangle = \langle \Delta_\Phi h_1,\, u_1 - u_2\rangle = 0$, and, similarly,
$\langle \Delta_\Phi h_2,\, h_1 - h_2\rangle = 0$.
By **Proposition 1**, we conclude that $h_1 - h_2 = u_1 - u_2 \in \mathcal{D}^\Phi(X)^G$.

Combining **Theorem 1** and **Theorem 2**, we obtain

Corollary

The space $\overline{H}^1(G, \ell^\Phi(X))$ *can be identified with* $\mathcal{HD}^\Phi(X)/\mathcal{D}^\Phi(X)^G$ *in a natural way.*

Combining **Theorem 1** and **Theorem 2**, we obtain

Corollary

The space $\overline{H}^1(G, \ell^\Phi(X))$ can be identified with $\mathcal{HD}^\Phi(X)/\mathcal{D}^\Phi(X)^G$ in a natural way.

As a result, we have

- $H^1(G, \ell^\Phi(X)) \cong \mathscr{D}^\Phi(X)/\ell^\Phi(X)$
- $\overline{H}^1(G, \ell^\Phi(X)) \cong \mathscr{D}^\Phi(X)/\overline{\ell^\Phi(X)}$

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- $H^1(G, \ell^\Phi(X)) \cong \mathscr{D}^\Phi(X)/\ell^\Phi(X)$
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and

- $\mathcal{D}^\Phi(X)/\mathcal{D}^\Phi(X)^G = ((\overline{\ell^\Phi(X)})_{\mathcal{D}^\Phi} \oplus \mathcal{HD}^\Phi(X))/\mathcal{D}^\Phi(X)^G$
- $\overline{H}^1(G, \ell^\Phi(X)) \cong \mathcal{HD}^\Phi(X)/\mathcal{D}^\Phi(X)^G$

Definition

A group G is *amenable* if there is a nontrivial finitely additive measure $\mu \colon 2^G \to \mathbb{R}^{\geq 0}$.

- The measure is *finitely additive*: given disjoint subsets A, $B \subseteq G$, the measure of the union of the sets is the sum of the measures
$$\mu(A \cup B) = \mu(A) + \mu(B)$$
.
- The measure is *left-invariant*: given a subset A and an element g of G, we have
$$\mu(L_g(A)) = \mu(A),$$
where $L_g \colon x \mapsto gx$.

Amenable groups

Amongst others, the following groups are amenable:

- Finite groups
- Compact groups
- Abelian groups
- Solvable groups

Amenable groups

The following assertion was obtained for locally compact groups by Ya. Kopylov (Amenability of closed subgroups and Orlicz spaces, *Sib. Élektron. Mat. Izv.*):

Proposition

Suppose that Φ is a Δ_2-regular N-function. If G is a countable group then the following are equivalent:
(i) $H^1(G, \ell^\Phi(G)) = \overline{H}^1(G, \ell^\Phi(G))$;
(ii) *G is not amenable.*

Normal Subgroups with Large Centralizer

Theorem 3

Let Φ be an N-function lying in $\Delta_2(0) \cap \nabla_2(0)$ and let N be a normal infinite finitely generated subgroup of a finitely generated group G. If N in non-amenable and its centralizer $Z_G(N)$ is infinite then $\overline{H}^1(G, \ell^\Phi(G)) = 0$.

The same argument that in the proof of **Theorem 3** yields the following

Corollary

If $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ and a finitely generated group G has infinite center then $H^1(G, \ell^\Phi(G)) = 0$.

A sufficient condition for the triviality of the first ℓ^Φ-cohomology of a wreath product of finitely generated groups.

Theorem 4

Suppose that G_1, G_2 are nontrivial finitely generated groups and $\Phi \in \Delta_2(0) \cap \nabla_2(0)$ and $G = G_1 \wr G_2$. If G_1 is nonamenable then $H^1(G, \ell^\Phi(G)) = 0$.

Thank you!