Classification of Complex Hyperbolic Isometries

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Knots, braids and automorphism groups
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Classification of isometries of $\mathbf{H}^2_{\mathbb{R}}$

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- An isometry f of $\mathbf{H}^2_{\mathbb{R}}$ is called elliptic if it has a fixed point on $\mathbf{H}^2_{\mathbb{R}}$. It is called parabolic, resp. hyperbolic if it is non-elliptic and has one, resp. two fixed points on the boundary $\partial\mathbf{H}^2_{\mathbb{R}} = \hat{\mathbb{R}} \approx \mathbb{S}^1$.

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(i) A acts as an elliptic isometry if and only if $tr^2(A) < 4$.
(ii) A acts as a parabolic isometry if and only if $tr^2(A) = 4$.
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- Our motivtion is to generalize this result for isometries of the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ where n arbitrary.

The Hermitian Space

- Equip $\mathbb{V} = \mathbb{C}^{n+1}$ with the Hermitian form $\langle z, w \rangle = -\overline{w}_0 z_0 + \overline{w}_1 z_1 + \cdots + \overline{w}_n z_n$.

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- Let $\mathbb{V}_- = \{z \in \mathbb{V} \mid \langle z, z\rangle < 0\}$, $\mathbb{V}_0 = \{z \in \mathbb{V} \mid \langle z, z\rangle = 0\}$, $\mathbb{V}_+ = \{z \in \mathbb{V} \mid \langle z, z\rangle > 0\}$.

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- A vector $v \in \mathbb{V}$ is called *time-like*, *space-like* or *light-like* according as v is an element in $\mathbb{V}_-$, $\mathbb{V}_+$ or $\mathbb{V}_0$.

The Complex Hyperbolic Space

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The metric d is complete.

The Ball Model

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- This identifies $\mathbf{H}^n_{\mathbb{C}}$ with the ball

$$\mathbf{B}^n_{\mathbb{C}} = \{\zeta = (\zeta_1, ..., \zeta_n) \mid \sum_{k=1}^{n} |\zeta_i|^2 < 1\}.$$

- From the ball model it is clear that the boundary of $\mathbf{H}^n_{\mathbb{C}}$ is the sphere

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- Here $Z(\mathrm{U}(n, 1)) = \{\lambda I \ : \ |\lambda| = 1\}$.
- For convenient, we often take $\mathrm{SU}(n, 1)$ as the group acting by the isometries.

Classification of isometries

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- Let g be an isometry of $\mathbf{H}^n_{\mathbb{C}}$. By Brouwer's fixed point theorem every element has a fixed point on $\overline{\mathbf{H}^n_{\mathbb{C}}} = \mathbf{H}^n_{\mathbb{C}} \cup \partial \mathbf{H}^n_{\mathbb{C}}$. If there is no fixed point on $\mathbf{H}^n_{\mathbb{C}}$, there can be at most two fixed points on the boundary.

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Definition

An eigenvalue λ (counted without multiplicity) of an unitary automorphism is said to be of *negative type*, of *positive type* or *null* if the corresponding eigenvector is in $\mathbb{V}_-$, $\mathbb{V}_+$ or $\mathbb{V}_0$ respectively.

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- For g loxodromic, it has exactly two null eigenvalues $re^{i\theta}$, $r^{-1}e^{i\theta}$, $r > 1$, $0 \leq \theta \leq 2\pi$. All other eigenvalues are positive and of norm 1.

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- g_u has minimal polynomial $(x-1)^2$ or $(x-1)^3$.

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Theorem

(Goldman) *Let* $f(t) = |t|^4 - 8\Re(t^3) + 18|t|^2 - 27$. *Let* $A \in \mathrm{SU}(2,1)$ *then:*

(i) A is loxodromic if and only if $f(tr(A)) > 0$.

(ii) A has a repeated eigenvalue if and only if $f(tr(A)) = 0$.

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- Here $f(t) = -R(\chi_A, \chi'_A)$, where R denotes the resultant.

Resultant

Recall that for

$$\begin{aligned} p(X) &= a_r X^r + a_{r-1} X^{r-1} + \cdots + a_1 X + a_0, \\ q(X) &= b_s X^s + b_{s-1} X^{s-1} + \cdots + b_1 X + b_0 \end{aligned}$$

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In the above case $q(X) = p'(X)$.

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- A is called k-loxodromic if it has k pairs of eigenvalues $r_j e^{i\theta_j}$ and $r_j^{-1} e^{i\theta_j}$ with $r_j > 1$ for $j = 1, \ldots, k$, and all other eigenvalues are unit complex numbers.

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- Also, A is said to be regular if the eigenvalues are mutually distinct, that is A has no repeated eigenvalues.

Theorem

Let $A \in \mathrm{SU}(p, q)$. *Let* $R(\chi_A, \chi'_A)$ *denotes the resultant of the characteristic polynomial* $\chi_A(X)$ *and its first derivative* $\chi'_A(X)$. *Then for* $m \geq 0$, *we have the following.*

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To prove the theorem, we use the fact that if $\alpha_1, \ldots \alpha_r$ are the roots (without multiplicities) of a degree r polynomial $p(x)$, then for a_r the leading coefficient.

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$$\begin{aligned} R(p, p') &= a_r^{r-1} \prod_{j=1}^{r} p'(\alpha_j) \\ &= a_r^{r-1} \prod_{j=1}^{r} a_r \prod_{i \neq j} (\alpha_j - \alpha_i) \end{aligned}$$

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R(p, p') &= a_r^{r-1} \prod_{j=1}^{r} p'(\alpha_j) \\
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Now, Write $p + q = n$. Suppose A is regular elliptic.
Then A has distinct eigenvalues

$$\lambda_1 = e^{i\theta_1},\ \lambda_2 = e^{i\theta_2},\ \dots,\ \lambda_n = e^{i\theta_n}.$$

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Since θ_j and θ_k are mutually distinct and $n(n-1)$ is always an even number, we must have $R(\chi_A, \chi_A') > 0$.

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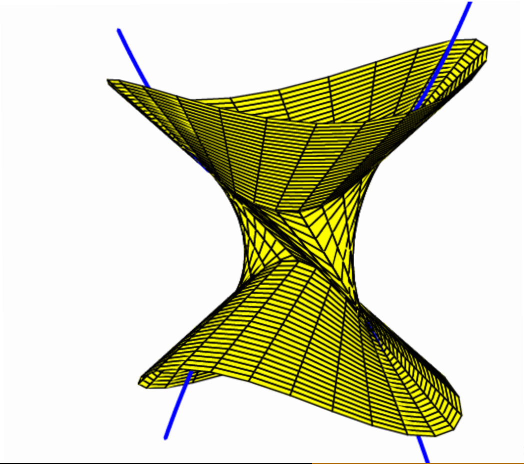

Classification for $\mathrm{SU}(p,q)$, $p+q=4$

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(i) *A is 2-loxodromic if and only if $R(\chi_A, \chi'_A) > 0$ and*

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(iv) A *has a repeated eigenvalue if and only if* $R(\chi_A, \chi'_A) = 0$.

Thank You!