R_∞ property for Chevalley groups

Timur Nasybullov

Novosibirsk State University

International Workshop "Knots, braids and automorphism groups"
Novosibirsk, Russia, July 21 – 25, 2014

Definitions

Let G be a group, φ be an automorphism of G

Definition 1

Two elements $x, y \in G$ are called (twisted) φ-conjugated if there exists an element $z \in G$, such that

$$x = zy\varphi(z^{-1}).$$

$x \sim_\varphi y$ – x and y are φ-conjugated
$[x]_\varphi$ – φ-conjugacy class of element x
$R(\varphi)$ – number of φ-conjugacy classes (Reidemeister number)

Example

Let $G = \mathbb{Z}$ be a free abelian group of rang 1
$\mathrm{Aut}\, G = \{id, \varphi\}$, where $\varphi : x \mapsto -x$

$$x \sim_{\varphi} y \Leftrightarrow x = z + y + \varphi(-z) = y + 2z \Rightarrow G = [0]_{\varphi} \cup [1]_{\varphi}, R(\varphi) = 2$$

$$x \sim_{id} y \Leftrightarrow x = z + y + id(-z) = y \Rightarrow G = \bigcup_{x \in G} [x]_{id}, R(id) = \infty$$

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

- Non-elementary Gromov hyperbolic groups (2001)

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

- Non-elementary Gromov hyperbolic groups (2001)

International Workshop "Knots, braids and automorphism groups"
Novosibirsk, Russia, July 21 – 25, 2014

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

- Non-elementary Gromov hyperbolic groups (2001)

International Workshop "Knots, braids and automorphism groups"
Novosibirsk, Russia, July 21 – 25, 2014

- Braid groups B_n, when $n \geq 3$ (2010)

Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

- Non-elementary Gromov hyperbolic groups (2001)

International Workshop "Knots, braids and automorphism groups"
Novosibirsk, Russia, July 21 – 25, 2014

- Braid groups B_n, when $n \geq 3$ (2010)
- Groups of non-trivial knots (2012)

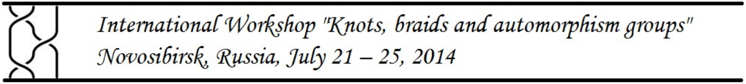
Problem

$R(\varphi)$ – number of φ-conjugacy classes

Question

For which groups the number $R(\varphi) = \infty$ for any automorphism φ? (R_∞ property)

- Non-elementary Gromov hyperbolic groups (2001)



- Braid groups B_n, when $n \geq 3$ (2010)
- Groups of non-trivial knots (2012)
- Non-amenable residually finite finitely generated groups (2012)

Problem

- $\mathrm{Sp}_{2n}(\mathbb{Z})$

Problem

- $\mathrm{Sp}_{2n}(\mathbb{Z})$

Question

For which linear groups the number $R(\varphi) = \infty$ for any automorphism φ?

Problem

- $\mathrm{Sp}_{2n}(\mathbb{Z})$

Question

For which linear groups the number $R(\varphi) = \infty$ for any automorphism φ?

- $\mathrm{SL}_n(K), \mathrm{GL}_n(K)$ for some rings K $(n \geq 3)$ (2012)

Problem

- $\mathrm{Sp}_{2n}(\mathbb{Z})$

Question

For which linear groups the number $R(\varphi) = \infty$ for any automorphism φ?

- $\mathrm{SL}_n(K), \mathrm{GL}_n(K)$ for some rings K ($n \geq 3$) (2012)
- $\mathrm{SL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z}), \mathrm{PSL}_n(\mathbb{Z}), \mathrm{PGL}_n(\mathbb{Z})$ (2012)

Results

Let G be a Chevalley group of type $\Phi \neq A_1$ over the field F

Results

Let G be a Chevalley group of type $\Phi \neq A_1$ over the field F

Theorem 1

If the automorphism group of field F is periodic, and the characteristic of F is equal to 0, then $R(\varphi) = \infty$ for any φ.

Results

Let G be a Chevalley group of type $\Phi \neq A_1$ over the field F

Theorem 1

If the automorphism group of field F is periodic, and the characteristic of F is equal to 0, then $R(\varphi) = \infty$ for any φ.

Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}(\theta)$

Results

Theorem 2

If F is a field of zero characteristic, such that that the transcendence degree of F over $\mathbb{Q}$ is finite, than
1) If $\Phi = A_l (l \geq 2),\ B_l (l \geq 2),\ E_8,\ F_4,\ G_2$, than G is R_∞-group.
2) If in F the equation $T^k = a$ can be solved for any $a \in F$, then G is R_∞-group for all other cases. Where

Φ	C_l	D_l	E_6	E_7
k	2	2	6	2

Results

Theorem 2

If F is a field of zero characteristic, such that that the transcendence degree of F over $\mathbb{Q}$ is finite, than
1) If $\Phi = A_l (l \geq 2),\ B_l (l \geq 2),\ E_8,\ F_4,\ G_2$, than G is R_∞-group.
2) If in F the equation $T^k = a$ can be solved for any $a \in F$, then G is R_∞-group for all other cases. Where

Φ	C_l	D_l	E_6	E_7
k	2	2	6	2

Examples: $\mathbb{Q},\ \mathbb{Q}(T_1,\dots,T_n),\ \overline{\mathbb{Q}},\ \overline{\mathbb{Q}(T_1,\dots,T_n)}$

Theorem 2

If F is an algebraically closed field of zero characteristic, such that that the transcendence degree of F over $\mathbb{Q}$ is finite, than G is R_∞-group.

Examples: $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}(T_1, \ldots, T_n)}$

Example

Theorem (R. Steinberg, 1968)

Let G be a connected linear algebraic group and φ an endomorphism of G onto G. If $|Fix_{\varphi}| < \infty$, then $[e]_{\varphi} = \{x\varphi(x^{-1}) \mid x \in G\} = G$.

Question

$[e]_\varphi$ – φ-conjugacy class of the unit element.

Proposition (A.Fel'shtyn, E.Troitsky)

The twisted conjugacy class $[e]_\varphi$ of the unit element e is a subgroup of abelian group G. The other ones are cosets $[g]_\varphi = g[e]_\varphi$.

Question

Question

When the twisted conjugacy class of the unit element is a subgroup?

Question

Question

When the twisted conjugacy class of the unit element is a subgroup?

Proposition

If φ is a central automorphism of group G, then $[e]_\varphi \trianglelefteq G$.

Results

Proposition

Let G be a Chevalley group of type $\Phi \neq A_1$ over the field F of zero characteristic. If F is such a field, that the group $\mathrm{Aut}(F)$ is periodic, or F is algebraically closed field, such that $\mathrm{tr.deg}_{\mathbb{Q}}(F) < \infty$, then $[e]_\varphi \leq G$ if and only if φ is a central automorphism.

Thats all, Thanks!