Cooper's problem and Nilpotent Group of Class 2

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- **The Cooper's problem (Problem no. 6.47 of KOUROVKA notebook)**: Let G be a group, v a group word in two variables such that the operation $x \circ y = v(x, y)$ defines the structure of a new group $G_v = (G, \circ)$ on the set G. Does G_v always lie in the variety generated by G?

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- Not Always.
- When?

Let us investigate the problem in case of Nilpotent group of class 2.

It is clear that any word in two variable in a nilpotent group G of class 2 can be written as $w(a,b) = a^k b^l [a,b]^m$, where $k, l, m \in \mathbb{Z}$ and $a, b \in G$

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$$e_1 \circ e = e = e \circ e_1.$$

This implies $e_1^k = e = e_1^l$.

Now $e_1 \circ e_1 = e_1$, that is $e_1^{l+k} = e_1$. Hence $e_1 = e$.

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Thus our operation becomes

$$a \circ b := ab[a, b]^m.$$

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For $a, b, c \in G$,

$$\begin{aligned} a \circ (b \circ c) &= ab[a,b]^m \circ c \\ &= abc[a,b]^m[ab,c]^m \\ &= abc[a,b]^m[a,c]^m[b,c]^m \\ &= (a \circ b) \circ c. \end{aligned}$$

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For this we need to give some definitions and very basic results.

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We call G to be *nilpotent group of class at most* 2 if it is either abelian or nilpotent group with class 2.

Definition

Let $(G,.)$ be a group of order r. An ordered $r-$ tuple $(n_1, n_2, \ldots, n_r)$ in $\mathbb{N}^r$ such that $1 = n_1 < n_2 \leq \ldots \leq n_r$ is called *order structure* of G if the elements of G can be put in a sequence $\{x_1, x_2, \ldots, x_r\}$ of length r such that the order of x_i in G is $n_i,\ 1 \leq i \leq r$.

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For example, the order structure of $C_2 \times C_4$ is $(1, 2, 2, 2, 4, 4, 4, 4)$.

Notation

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- For $x \in A$, we denote $|x|$ for the order of x in A.
- Further for $a, b \in \mathbb{N}$, the positive least common multiple of a and b is denoted by $[a, b]$.
- For $x, y \in G$, we denote $[x, y]$ for $x^{-1}y^{-1}xy$.

Lemma

Let $G = P_1 \times P_2 \times \ldots \times P_s$ and $H = Q_1 \times Q_2 \times \ldots \times Q_s$ be two groups such that P_i and Q_i have same order structures, $1 \leq i \leq s$. Then the order structures of G and H are same.

Proof.

By induction, it is sufficient to prove the lemma for $s = 2$. Let $(n_1, n_2, \ldots, n_{r_1})$ and $(m_1, m_2, \ldots, m_{r_2})$ denote the order structures of P_1 and P_2 respectively. Then we can write $P_1 = \{x_1, \ldots, x_{r_1}\},\ Q_1 = \{x'_1, \ldots, x'_{r_1}\},\ P_2 = \{y_1, \ldots, y_{r_2}\}$ and $Q_2 = \{y'_1, \ldots, y'_{r_2}\}$ such that $|x_i| = |x'_i| = n_i,\ 1 \leq i \leq r_1$ and $|y_j| = |y'_j| = m_j, 1 \leq j \leq r_2$. Since $|(x_i, y_j)| = |(x'_i, y'_j)| = [n_i, m_j]$, the lemma follows. $\square$

Corollary

Two finite abelian groups are isomorphic if and only if they have same order structures.

Proof.

Follows from Lemma and the structure theorem for finite abelian groups. $\square$

Lemma

Let G be a nilpotent group with class 2. Let $x, y, z \in G$. Then
(i) $[xy, z] = [x, z][y, z]$*,*
(ii) $[x^k, y] = [x, y^k] = [x, y]^k$*, where k is an integer.*

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- $(((x^{-1} \circ^n y^{-1}) \circ^n x) \circ^n y) = [x, y]^{2n+1}$.
- If x is in center of G, then $x^{-1} \circ^n y^{-1} = y^{-1} \circ^n x^{-1}$ for all $y \in G$
- $(G, \circ^n)$ is a group of nilpotent class at most 2.

Definition

Given a nilpotent group $(G, .)$ with class at most 2 and $n \in \mathbb{N}$, we define another group $(S_n(G), \circ^n)$ as follows: $S_n(G) = G$, for $x, y \in S_n(G)$, $x \circ^n y := [x, y]^n xy$.

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Further, we define a sequence $\{S_n^i(G)\}_i$ by induction. Let $S_n^0(G) = G$. If $S_n^i(G)$ is defined, then $S_n^{i+1}(G) = S_n(S_n^i(G))$.

Lemma

Let G be a nilpotent group with class 2 and $n \in \mathbb{N}$. Then $S_n^i(G) = S_{s(i)}(G)$, where $s(i) = \frac{(2n+1)^i - 1}{2}$.

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Lemma

Let G be a nilpotent group with class 2 and $|G| = p^k$, where p is an odd prime. Then for $n = \frac{p-1}{2}$, we have

(i) $x \in Z(S_n^i(G)) \iff x^{p^i} \in Z(G)$.

(ii) $x \in Z(S_n^{i+1}(G)) \iff x^p \in Z(S_n^i(G))$.

(iii) $Z(S_n^i(G))$ is a normal subgroup of $S_n^{i+1}(G)$.

Let $r = p^k$, where p is an odd prime and $k \in \mathbb{N}$. Let X^r denotes a complete set of non-isomorphic nilpotent groups with class at most 2 and order r. We say two elements of X^r are related if both have same order structure. This relation is an equivalence relation. Let $X_1^r, X_2^r, \ldots, X_l^r$ be the distinct equivalence classes of the set X^r under this relation.

Let G be a nilpotent group with class at most 2 and order r. Identify G with its unique isomorphic copy in X^r. We may assume that $G \in X_1^r$.
Take $n = \frac{p-1}{2}$. Suppose that the exponent of the commutator subgroup of G is p^t. We define the *string of groups* associated with G to be a sequence $F_i(G)$ of length $t+1$ in X_1^r as $F_i(G) \cong S_n^i(G),\ 0 \leq i \leq t$.

Theorem

(a) No two terms of the string of a group G are isomorphic.
(b) The last term of the string of a group G is an abelian group.
(c) The last terms of the string of any two members of X_i^r are same.

Proof of the Theorem

Let p be an odd prime and G be a nilpotent group of order $r = p^k$ with class 2. Let the exponent of the commutator subgroup of G be p^t. Assume that $G \in X_1^r$. Identify $F_i(G)$ with $S^i_{\frac{p-1}{2}}(G), 0 \leq i \leq t$. Take $n = \frac{p-1}{2}$.

Proof of the Theorem

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(a) To prove it we will show that the centers of any two distinct terms of the string of G are not equal. Fix $i \in \{0, \ldots, t-1\}$. By Lemma 8 (iii), $Z(S_n^i(G))$ is a normal subgroup of $S_n^{i+1}(G)$. Let $H = S_n^{i+1}(G)/Z(S_n^i(G))$. We claim that $|H| > 1$. Suppose that $S_n^{i+1}(G) = Z(S_n^i(G))$. Then $S_n^i(G) = Z(S_n^i(G))$. By Lemma 8(i), $x^{p^i} \in Z(G)$ for all $x \in G$. Thus the exponent of $G/Z(G)$ is less than equal to p^i.

Proof of the Theorem

By Lemma, it follows that the exponent of $G/Z(G)$ is equal to the exponent of the commutator subgroup of G. This is a contradiction for the exponent of the commutator subgroup of G is p^t.

Take $\Omega_1(H) = \langle \{\bar{x} \in H | \bar{x}^p = \bar{1} \in H\} \rangle$, where $\bar{1}$ denotes the identity element of H. But since H is nilpotent, $\Omega_1(H) \neq \{\bar{1}\}$. This implies that there exists $x \in S_n^{i+1}(G)$ such that $x \notin Z(S_n^i(G))$ and $x^p \in Z(S_n^i(G))$. By Lemma 8(ii), $Z(S_n^i(G)) \subsetneq Z(S_n^{i+1}(G))$ for all $0 \leq i \leq t-1$. That is, $Z(G) \subsetneq Z(S_n(G)) \subsetneq Z(S_n^2(G)) \subsetneq \ldots \subsetneq Z(S_n^t(G))$. Hence distinct terms of the string of G are non-isomorphic groups.

Proof of the Theorem

(b) The last term of the string of G is $S_n^t(G)$. By Lemma , $S_n^t(G) = S_{s(t)}(G)$ where $s(t) = \frac{(2n+1)^t - 1}{2}$. But $n = \frac{p-1}{2}$, so $s(t) = \frac{p^t - 1}{2}$. Now let $x, y \in S_n^t(G)$. Then

$$
\begin{aligned}
x \circ^{s(t)} y &= [x,y]^{s(t)} xy \\
&= [x,y]^{\frac{p^t-1}{2}} xy \\
&= [x,y]^{p^t - \frac{p^t+1}{2}} xy \\
&= [x,y]^{-\frac{p^t+1}{2}} xy \qquad \text{(for } p^t \text{ is the exponent of the commutator)} \\
&= [y,x]^{\frac{p^t+1}{2}} xyx^{-1}y^{-1}yx \qquad \text{(for } [x,y]^{-1} = [y,x]) \\
&= [y,x]^{\frac{p^t+1}{2}} [x^{-1}, y^{-1}]yx \\
&= [y,x]^{\frac{p^t+1}{2}} [y,x]^{-1}yx \\
&= [y,x]^{\frac{p^t-1}{2}} yx \\
&= y \circ^{s(t)} x.
\end{aligned}
$$

So, $S_n^t(G)$ is an abelian group.

Proof of the Theorem

(c) Let $G, H \in X_i^r$. Since the last term of the string is an abelian group and for a given order structure there is unique (up to isomorphism) abelian group (Corollary), so last term of G and H are same.

Corollary

Let $(G, .)$ be an odd order nilpotent group with class 2 such that the exponent of the commutator subgroup is $p_1^{r_1} p_2^{r_2} \dots p_s^{r_s}$, where p_i's are the distinct prime divisors of the order of G and r_i's are non-negative integers. Then there are at least $\left(\prod_{i=1}^{s}(1 + r_i)\right) - 1$ non-isomorphic nilpotent groups with class 2 and each of the groups has the same order structure as G.

Proof.

Since G is nilpotent, $G = P_1 \times P_2 \times \ldots \times P_s$, where P_j is the Sylow p_j-subgroup of G, $1 \leq j \leq s$. Let $F(P_j) := \{F_i(P_j) | 0 \leq i \leq r_j\}$, where $F_i(P_j)$ denotes the $(i+1)$-th term of the string of P_j. Let $T = \left\{ \prod_{j=1}^{s} F_j | F_j \in F(P_j) \right\}$. Then by Lemma 3, the elements of T have same order structure. Also by Theorem 9 (a), elements of T are non-isomorphic groups. We observe that $G = \prod_{j=1}^{s} F_0(P_j) \in T$. Further, $H = \prod_{j=1}^{s} F_{r_j}(P_j)$ is the only element in T which is an abelian group (Theorem 9 (b)). Thus there are $|T| - 1 = \left(\prod_{i=1}^{s} (1 + r_i) \right) - 1$ non-isomorphic nilpotent groups with class 2 of the same order structure as G. $\square$

take $r = p^4$, where p is an odd prime, then the following is a complete set of non-isomorphic nilpotent groups with class 2 and order r.

(i) $A = \langle x, y | x^{p^3} = 1, y^p = 1, y^{-1}xy = x^{1+p^2} \rangle$,

(ii) $B = \langle x, y, z | x^{p^2} = 1, y^p = 1, z^p = 1, z^{-1}yz = yx^p, y^{-1}xy = x, z^{-1}xz = x \rangle$,

(iii) $C = \langle x, y | x^{p^2} = 1, y^{p^2} = 1, y^{-1}xy = x^{1+p} \rangle$,

(iv) $D = \langle x, y, z | x^{p^2} = 1, y^p = 1, z^p = 1, z^{-1}xz = x^{1+p}, x^{-1}yx = y, z^{-1}yz = y \rangle$,

(v) $E = \langle x, y, z | x^{p^2} = 1, y^p = 1, z^p = 1, z^{-1}xz = xy, y^{-1}xy = x, z^{-1}yz = y \rangle$,

(vi) $F = \langle x, y, z, a, | x^p = 1, y^p = 1, z^p = 1, a^p = 1, a^{-1}za = zx, a^{-1}ya = y, a^{-1}xa = x, z^{-1}yz = y, z^{-1}xz = x, y^{-1}xy = x \rangle$.

Then we may take $X^r = \{A, B, C, D, E, F, C_{p^3} \times C_p, C_{p^2} \times C_p \times C_p, C_{p^2} \times C_{p^2}, C_p \times C_p \times C_p \times C_p, C_{p^4}\}$, where C_n denotes the cyclic group of order n. Equivalence classes of X^r with respect to order structure relation are $X_1^r = \{A, C_{p^3} \times C_p\}$, $X_2^r = \{B, D, E, C_{p^2} \times C_p \times C_p\}$, $X_3^r = \{C, C_{p^2} \times C_{p^2}\}$, $X_4^r = \{F, C_p \times C_p \times C_p \times C_p\}$ and $X_5^r = \{C_{p^4}\}$. In X_2^r, the commutator subgroups of B, D and E have exponent p, so the length of the string of each of these groups will be 2.

Application

Let G be a nilpotent group with class 2 and order p^k where p is an odd prime. Let p^e be the exponent of commutator of G. Let $G = F_0(G), F_1(G), \ldots, F_e(G)$ be the string associated with G. Then

$$Aut(F_0(G)) \subseteq Aut(F_1(G)) \subseteq \ldots \subseteq Aut(F_e(G)).$$

- Search counter examples for Cooper's problem in case of nilpotent group of class 2.
- What are the subgroups of the automorphism group of finite abelian p-group G (say) (where p is odd prime), which can be an automorphism group of some nilpotent group of class 2 of order $|G|$?
- After recognizing such subgroups can we construct Nilpotent p group of class 2 for which it is an automorphism group?

Thank You