On Vassiliev invariants of braids of the sphere

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"Knots, braids and automorphism groups ",
Novosibirsk, July 22, 2014

The talk is based on the joint work with Nizar Kaabi (Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia, 2012).

Plan

Introduction

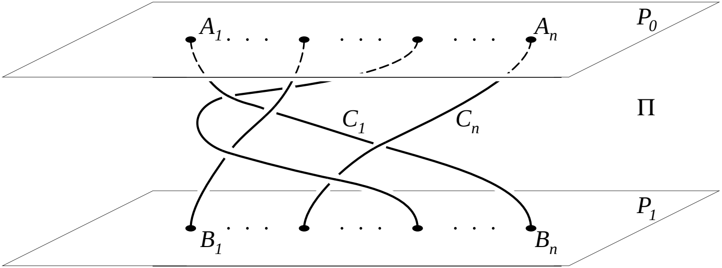
The theory of Vassiliev (or finite type) invariants starts with the works of V. A. Vasiliev though the ideas which lie in the foundations of this theory can be found in the work of M. Gousarov.The basic idea is classical in Mathematics: to introduce a filtration in a complicated fundamental object such that the corresponding associated graded object is simpler and sometimes possible to describe.

Braids

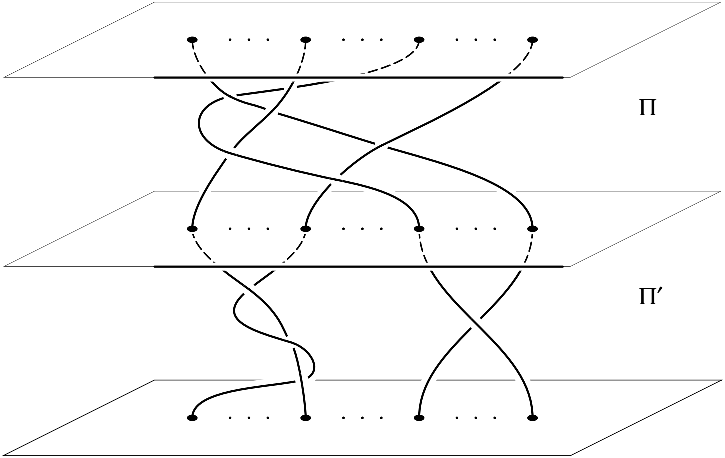
We remind the geometrical definition of braids. Let us consider two parallel planes P_0 and P_1 in $\mathbb{R}^3$, which contain two ordered sets of points A_1, ..., $A_n \in P_0$ and B_1, ..., $B_n \in P_1$. These points are lying on parallel lines L_A and L_B respectively. The space between the planes P_0 and P_1 we denote by Π.

Let us connect the set of points A_1, ..., A_n with the set of points B_1, ..., B_n by simple nonintersecting curves C_1, ..., C_n lying in the space Π and such that each curve meets only once each parallel plane P_t lying in the space Π.
This object is called a *geometric braid* and the curves are called the *strings* of a geometric braid.
Two geometric braids β and β' on n strings are *isotopic* if β can be continuously deformed into β' in the class of braids.
The relation of isotopy is an equivalence relation on the class of geometric braids on n strings. The corresponding equivalence classes are called *braids on n strings*.

Braids



On the set Br_n of braids the structure of a group introduces as follows.



Braids

Unit element is the equivalence class containing a braid of n parallel intervals, the braid β^{-1} inverse to β is defined by reflection of β with respect to the plane $P_{1/2}$. A string C_i of a braid β connects the point A_i with the pont B_{k_i} defining the permutation S^{β}. If this permutation is identical then the braid β is called *pure*.
The subgroup of pure braids for a manifold M is usually denoted $P_n(M)$.

Braid groups of the sphere and the mapping class groups of the sphere with n punctures

Let M be a topological space and let M^n be the n-fold Cartesian product of M. The *n-th ordered configuration space* $F(M,n)$ is defined by

$$F(M,n) = \{(x_1,\dots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with subspace topology of M^n. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The orbit space

$$B(M,n) = F(M,n)/\Sigma_n$$

is called the *n-th unordered configuration space* or simply *n-th configuration space*.

Braid groups of the sphere and the mapping class groups of the sphere with n punctures

The *braid group* $B_n(M)$ is defined to be the fundamental group $\pi_1(B(M,n))$. The *pure braid group* $P_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n)$. From the covering $F(M,n) \to F(M,n)/\Sigma_n$, we get a short exact sequence of groups

$$\{1\} \to P_n(M) \to B_n(M) \to \Sigma_n \to \{1\}. \tag{1}$$

Braid groups of the sphere and the mapping class groups of the sphere with n punctures

We will use later the following classical Fadell-Neuwirth Theorem.

Theorem

For $n > m$ the coordinate projection (forgetting of $n - m$ coordinates)

$$\delta_m^{(n)}: F(M, n) \to F(M, m), \ (x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$$

is a fiber bundle with fiber $F(M \setminus Q_m, n - m)$, where Q_m is a set of m distinct points in M.

In this work we consider the case $M = S^2$ and classical braids which are braids of the disc: $M = D^2$.

Braid groups of the sphere and the mapping class groups of the sphere with n punctures

Usually the braid group of the disc $Br_n = B_n(D^2)$ is given by the following Artin presentation. It has the generators σ_i, $i = 1, ..., n-1$, and two types of relations:

$$\begin{cases} \sigma_i\sigma_j & = \sigma_j\,\sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i & = \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \end{cases} \tag{2}$$

The generators $a_{i,j}$, $1 \le i < j \le n$ of the pure braid group P_n (of a disc) can be described as elements of the braid group Br_n by the formula:

$$a_{i,j} = \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}.$$

The defining relations among $a_{i,j}$, which are called the *Burau relations* are as follows:

$$\begin{cases} a_{i,j} a_{k,l} = a_{k,l} a_{i,j} \text{ for } i < j < k < l \text{ and } i < k < l < j, \\ a_{i,j} a_{i,k} a_{j,k} = a_{i,k} a_{j,k} a_{i,j} \text{ for } i < j < k, \\ a_{i,k} a_{j,k} a_{i,j} = a_{j,k} a_{i,j} a_{i,k} \text{ for } i < j < k, \\ a_{i,k} a_{j,k} a_{j,l} a_{j,k}^{-1} = a_{j,k} a_{j,l} a_{j,k}^{-1} a_{i,k} \text{ for } i < j < k < l. \end{cases} \tag{3}$$

It was proved by O. Zariski and then rediscovered by E. Fadell and J. Van Buskirk that a presentation of the braid group on a 2-sphere can be given with the generators σ_i, $i = 1, ..., n-1$, the same as for the classical braid group, satisfying the braid relations (2) and the following sphere relation:

$$\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \dots \sigma_2\sigma_1 = 1. \tag{4}$$

Let Δ be the Garside's fundamental element in the braid group Br_n. It can be expressed in particular by the following word in canonical generators:

$$\Delta = \sigma_1 \dots \sigma_{n-1}\sigma_1 \dots \sigma_{n-2} \dots \sigma_1\sigma_2\sigma_1.$$

Braids

Braid groups of the sphere and the mapping class groups of the sphere with n punctures

For the pure braid group on a 2-sphere let us introduce the elements $a_{i,j}$ for all i, j by the formulas:

$$\begin{cases} a_{j,i} = a_{i,j} & \text{for } i < j \leq n, \\ a_{i,i} = 1. \end{cases} \tag{5}$$

The pure braid group on a 2-sphere has the generators $a_{i,j}$ which satisfy the Burau relations (3), the relations (5), and the following relations:

$$a_{i,i+1} a_{i,i+2} \dots a_{i,i+n-1} = 1 \quad \text{for all } i \leq n,$$

with the convention that indices are considered $\mod n$: $k + n = k$.

Let $S_{g,b,n}$ be an oriented surface of genus g with b boundary components and we remind that Q_n denotes a set of n punctures (marked points) in the surface. Consider the group $\mathrm{Homeo}(S_{g,b,n})$ of orientation preserving self-homeomorphisms of $S_{g,b,n}$ which fix pointwise the boundary (if $b > 0$) and map the set Q_n into itself. Let $\mathrm{Homeo}^0(S_{g,b,n})$ be the normal subgroup of self-homeomorphisms of $S_{g,b,n}$ which are isotopic to identity. Then the *mapping class group* $M_{g,b,n}$ is defined as a quotient group

$$M_{g,b,n} = \mathrm{Homeo}(S_{g,b,n}) / \mathrm{Homeo}^0(S_{g,b,n}).$$

Like braid groups the groups $M_{g,b,n}$ has a natural epimorphism to the symmetric group Σ_n with the kernel called the *pure mapping class group* $PM_{g,b,n}$, so there exists an exact sequence:

$$1 \to PM_{g,b,n} \to M_{g,b,n} \to \Sigma_n \to 1.$$

Geometrically the pure mapping class group $PM_{g,b,n}$ consists of isotopy classes of homeomorphisms that preserve the punctures pointwise.

Consider the pure mapping class group $PM_{0,0,n}$ of a punctured 2-sphere (so the genus is equal to 0) with no boundary components that we simply denote by PM_n; the same way we denote further $M_{0,0,n}$ simply by M_n.
The group PM_n is closely related to the pure braid group $P_n(S^2)$ on the 2-sphere as well as its non-pure analogue M_n is related with the (total) braid group $B_n(S^2)$ on the 2-sphere.
W. Magnus obtained a presentation of the mapping class group M_n for the n-punctured 2-sphere. It has the same generators as $B_n(S^2)$ and a complete set of relations consists of (2), (4) and the following relation

$$(\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1})^n = 1. \tag{6}$$

This defines an epimorphism

$$\gamma : B_n(S^2) \to M_n.$$

Theorem

(i) The pure braid group on a 2-sphere $P_n(S^2)$, $n \geq 3$ is isomorphic to the direct product of the cyclic group C_2 of order 2 and PM_n.
(ii) The pure braid group P_n, $n \geq 2$ is isomorphic to the direct product of the infinite cyclic group C and PM_{n+1}.
(iii) The groups PM_n and $P_{n-3}(S^2_3)$ are isomorphic for $n \geq 4$.
(iv) There is a commutative diagram of groups and homomorphisms

$$\begin{array}{ccc} P_n \cong & PM_{n+1} \times C \\ \Big\downarrow \rho_p & \Big\downarrow \delta \times \rho \\ P_n(S^2) \cong & PM_n \times C_2, \end{array} \tag{7}$$

where ρ_p is the canonical epimorphism $P_n \to P_n(S^2)$, δ is induced by the Fadell-Neuwirth fibration via the isomorphism of item (iii) and ρ is the canonical epimorphism of the infinite cyclic group onto the cyclic group of order 2.

Lie algebras obtained from the filtration of descending central series of the pure braid groups are essential ingredients in the construction of the universal Vassiliev invariant. We describe such Lie algebras for the groups $P_n(S^2)$ and $\mathcal{PM}_{0,n}$.We will use them in the next section in our construction of universal invariant.
For a group G the descending central series

$$G = \Gamma_1 > \Gamma_2 > \cdots > \Gamma_i > \Gamma_{i+1} > \dots .$$

is defined by the formula

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

This series of groups gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $gr_\Gamma^*(G)$

$$gr_\Gamma^i(G) = \Gamma_i/\Gamma_{i+1}.$$

A presentation of the Lie algebra $gr^*_\Gamma(P_n)$ for the pure braid group was done in the work of T. Kohno, and can be described as follows. It is the quotient of the free Lie algebra $L[A_{i,j} | 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given as follows:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{8}$$

It is convenient sometimes to have conventions like (5). So let us introduce the generators $A_{i,j}$, $1 \le i, j \le n$, not necessary $i < j$, by the formulae

$$\begin{cases} A_{j,i} = A_{i,j} \ \text{ for } \ 1 \le i < j \le n, \\ A_{i,i} = 0 \ \text{ for all } \ 1 \le i \le n. \end{cases}$$

For this set of generators the defining relations (8) can be rewritten as follows

$$\begin{cases} A_{i,j} = A_{j,i} \ \text{ for } \ 1 \le i, j \le n, \\ A_{i,i} = 0 \ \text{ for } \ 1 \le i \le n, \\ [A_{i,j}, A_{s,t}] = 0, \ \text{if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \ \text{ for all different } \ i, j, k. \end{cases} \tag{9}$$

Y. Ihara gave a presentation of the Lie algebra $gr^*_\Gamma(P_n(S^2))$ of the pure braid group of a sphere. It is the quotient of the free Lie algebra $L[B_{i,j}| 1 \leq i,j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i,j \leq n$ modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \ \text{ for } 1 \leq i,j \leq n, \\ B_{i,i} = 0 \ \text{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \text{if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^n B_{i,j} = 0, \ \text{for } 1 \leq i \leq n. \end{cases} \tag{10}$$

It is also a quotient algebra of the Lie algebra $gr^*_\Gamma(P_n)$: the relations of the last type in (9) are the consequences of the third and the forth type relations in (10).

Theorem

The graded Lie algebra $gr^*_\Gamma(PM_n)$ *is the quotient of the free Lie algebra* $L[B_{i,j}|\, 1 \leq i, j \leq n]$ *modulo the following relations:*

$$\begin{cases} B_{i,j} = B_{j,i} \ \textit{for} \ 1 \leq i, j \leq n, \\ B_{i,i} = 0 \ \ \textit{for} \ 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \textit{if} \ \{i, j\} \cap \{s, t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \ \textit{for} \ 1 \leq i \leq n, \\ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{i,j} = 0. \end{cases} \tag{11}$$

Theorem

The graded Lie algebra $gr^*_\Gamma(PM_n)$ *is the quotient of the free Lie algebra* $L[B_{i,j}|\,1 \le i,j \le n-1]$ *generated by the elements* $B_{i,j}$, $1 \le i,j \le n-1$, *(smaller number of generators than in (i)) modulo the following relations:*

$$\begin{cases} B_{i,j} - B_{j,i} = 0 \ \ \textit{for} \ \ 1 \le i,j \le n-1, \\ B_{i,i} = 0 \ \ \textit{for} \ \ 1 \le i \le n-1, \\ [B_{i,j}, B_{s,t}] = 0, \ \ \textit{if} \ \ \{i,j\} \cap \{s,t\} = \emptyset, \\ [B_{i,j}, B_{i,k} + B_{j,k}] = 0 \ \ \textit{for all different} \ \ i,j,k, \\ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} B_{i,j} = 0. \end{cases} \tag{12}$$

We sketch briefly the basic ideas of the theory of Vassiliev invariants for braids.
Let A be an abelian group, then the group V of all maps (non necessary homomorphisms) from $B_n(S^2)$ to A is called the group of *invariants* of $B_n(S^2)$:

$$V = \mathrm{Map}(B_n(S^2), A).$$

If A is a commutative ring then V becomes an A-module.
Let $\mathbb{Z}[B_n(S^2)]$ be the group ring of the group $B_n(S^2)$, then

$$\mathrm{Map}(B_n(S^2), A) = \mathrm{Hom}(\mathbb{Z}[B_n(S^2)], A).$$

where $\mathrm{Hom}(\mathbb{Z}[B_n(S^2)], A)$ is an abelian group of homomorphisms of the group $\mathbb{Z}[B_n(S^2)]$ into the group A.

We can enlarge an invariant $v \in V$ for singular braids using the rule

$$v(\text{singular crossing of } i\text{-th and } i+1 \text{ strands}) = v(\sigma_i) - v(\sigma_i^{-1}).$$

The elements $\sigma_i - \sigma_i^{-1} \in \mathbb{Z}[B_n(S^2)]$, $i = 1, \dots, n-1$, generate an ideal of the ring $\mathbb{Z}[B_n(S^2)]$ which we denote by W; degrees of this ideal define a multiplicative filtration (*Vassiliev filtration*) $W^m = \Phi^m(\mathbb{Z}[B_n(S^2)])$. An invariants $v \in V$ is called *of degree m* if $v(x) = 0$ for all $x \in \Phi^{m+1}(\mathbb{Z}[B_n(S^2)])$. So the group V_m of invariants of degree m is defined as

$$V_m = \mathrm{Hom}(\mathbb{Z}[B_n(S^2)]/\Phi^{m+1}(\mathbb{Z}[B_n(S^2)]), A).$$

The advantage of braids is that this filtration can be characterized completely algebraically. Let S be a map from the symmetric group Σ_n:

$$S : \Sigma_n \to B_n(S^2)$$

which is a section of the canonical epimorphism $B_n(S^2) \to \Sigma_n$ (1). It is not a homomorphism which does not exist with such a condition. We can set up, for example, $S(s_i) = \sigma_i$. A similar map

$$S_M : \Sigma_n \to M_n$$

is a section of the canonical epimorphism $M_n \to \Sigma_n$.

Let I be the augmentation ideal of the group ring $\mathbb{Z}[P_n(S^2)]$. The powers of I generate a filtration of the ring $\mathbb{Z}[P_n(S^2)]$ and hence of the ring $\mathbb{Z}[P_n(S^2)] \otimes \mathbb{Z}[\Sigma_n]$ as we assume that elements of $\mathbb{Z}[\Sigma_n]$ have zero filtration. The same filtration we have in $\mathbb{Z}[M_n]$.

Proposition

There is an isomorphism of abelian groups with filtration

$$\mathbb{Z}[P_n(S^2)] \otimes \mathbb{Z}[\Sigma_n] \cong \mathbb{Z}[B_n(S^2)],$$

$$\mathbb{Z}[PM_n] \otimes \mathbb{Z}[\Sigma_n] \cong \mathbb{Z}[M_n],$$

which are induced by the canonical inclusions of the pure groups and the maps S and S_M; the rings $\mathbb{Z}[B_n(S^2)]$ and $\mathbb{Z}[M_n]$ are equipped with Vassiliev filtration.

Let c be the generator of the infinite cyclic group C and let $\mathbb{Z}[C]$ be the group ring of C. We denote by C_2 the cyclic group of the order 2 with the generator a, $\mathbb{Z}[C_2]$ is the group ring of C_2 and we define the homomorphism

$$\rho : \mathbb{Z}[C] \to \mathbb{Z}[C_2],$$

by the formula

$$\rho(c) = a.$$

Proposition

There are isomorphisms of rings

$$\mathbb{Z}[P_n] \cong \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C],$$

$$\mathbb{Z}[P_n(S^2)] \cong \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2],$$

which can be included into the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[P_n] \arrow[d, "\rho_p"] \arrow[r, phantom, "\cong"] & \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \arrow[d, "\delta\otimes\rho"] \\
\mathbb{Z}[P_n(S^2)] \arrow[r, phantom, "\cong"] & \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2],
\end{tikzcd}$$

where the morphisms ρ_p and δ in the diagram are induced by the corresponding morphisms of the diagram (7).

Proposition

The intersections of Vassiliev filtration for $\mathbb{Z}[B_n(S^2)]$ and $\mathbb{Z}[M_n]$ are trivial

$$\bigcap_{m\geq 0} \Phi^m(\mathbb{Z}[B_n(S^2)]) = 0, \qquad \bigcap_{m\geq 0} \Phi^m(\mathbb{Z}[M_n]) = 0.$$

The groups $\Phi^m(\mathbb{Z}[M_n])/\Phi^{m+1}(\mathbb{Z}[M_n])$ are torsion free. The group PM_n is residually torsion free nilpotent, the group $P_n(S^2)$ is residually nilpotent.

The filtered algebra $\mathcal{P}_n$ is defined as the universal enveloping algebra of the Lie algebra $gr_\Gamma^*(P_n)$ for the standard pure braid group

$$\mathcal{P}_n = U(gr_\Gamma^*(P_n)).$$

Its completion $\widehat{\mathcal{P}_n}$ is the target of the universal Vassiliev invariant for the pure braids

$$\mu : \mathbb{Z}[P_n] \to \widehat{\mathcal{P}_n}.$$

Let $\mathcal{PM}_n$ be the universal enveloping algebra of the Lie algebra $gr_\Gamma^*(PM_n)$; so as an associative algebra it has the generators which are in one-to-one correspondence with the generators $B_{i,j}$ of $gr_\Gamma^*(PM_n)$, say it will be $x_{i,j}$, $1 \leq i, j \leq n$, which satisfy the associative form of relations (12). Also we denote by $\mathcal{P}_n(S^2)$ the universal enveloping algebra of the Lie algebra $gr_\Gamma^*(P_n(S^2))$.

As usual one can define a Hausdorff filtration (intersection is zero) on $\mathcal{PM}_n$ and on $\mathcal{P}_n(S^2)$ by giving a degree 1 to each generator $x_{i,j}$. The canonical epimorphism of groups $\rho_p : P_n \to P_n(S^2)$ induces an epimorphism of filtered algebras

$$\rho_a : \mathcal{P}_n \to \mathcal{P}_n(S^2).$$

We denote by $\widehat{\mathcal{PM}}_n$ the completion of $\mathcal{PM}_n$ with respect to the topology, defined by this filtration. The same way $\widehat{\mathcal{P}_n}(S^2)$ is the completion of $\mathcal{P}_n(S^2)$. The algebra $\widehat{\mathcal{PM}}_n$ can be also described as an algebra of non-commutative power series of $x_{i,j}$ factorized by the closed ideal generated by the left hand sides of relations (12).

Let $\widehat{\mathcal{A}}$ be an associative algebra with unit such that as an abelian group it is isomorphic to the direct sum of integers and 2-adic numbers $\mathbb{Z} \oplus \mathbb{Z}_2$. We denote the generator of the first summand by 1 and the generator of the second summand by x. The multiplication in $\widehat{\mathcal{A}}$ is given by the rule

$$x^2 = -2x.$$

This algebra is filtered as follows $\Phi^0 = \widehat{\mathcal{A}}$, $\Phi^1 = \mathbb{Z}_2$, Φ^m is generated by $2^m x$, for $m = 2, 3 \ldots$.

We define the homomorphisms

$$\alpha : \mathbb{Z}[C_2)] \to \widehat{\mathcal{A}},$$

$$\chi : \mathbb{Z}[C] \to \mathbb{Z}[[y]],$$

$$\beta : \mathbb{Z}[[y]] \to \widehat{\mathcal{A}}$$

by the formulae

$$\alpha(a) = 1 + x, \quad \chi(c) = 1 + y, \quad \beta(y) = x. \tag{13}$$

Proposition

The homomorphisms of rings α and χ respect the filtration and induce a multiplicative isomorphism at the associated graded level. They fit in the following commutative diagram of homomorphisms of rings.

$$\begin{tikzcd}
\mathbb{Z}[C] \arrow[r, "\rho"] \arrow[d, "\chi"] & \mathbb{Z}[C_2] \arrow[d, "\alpha"] \\
\mathbb{Z}[[y]] \arrow[r, "\beta"] & \widehat{\mathcal{A}}.
\end{tikzcd}$$

Proposition

There are isomorphisms of filtered rings

$$\widehat{\mathcal{P}}_n \cong \widehat{\mathcal{PM}}_{n+1}\widehat{\otimes}\mathbb{Z}[[y]],$$

$$\widehat{\mathcal{P}}_n(S^2) \cong \widehat{\mathcal{PM}}_n\widehat{\otimes}\widehat{\mathcal{A}},$$

which can be included into the following commutative diagram of filtered ring homomorphisms

$$\begin{tikzcd}
\widehat{\mathcal{P}}_n \arrow[r, phantom, "\cong"] \arrow[d, "\widehat{\rho}_a"] & \widehat{\mathcal{PM}}_{n+1}\widehat{\otimes}\mathbb{Z}[[y]] \arrow[d, "\widehat{\delta}\widehat{\otimes}\beta"] \\
\widehat{\mathcal{P}}_n(S^2) \arrow[r, phantom, "\cong"] & \widehat{\mathcal{PM}}_n\widehat{\otimes}\widehat{\mathcal{A}},
\end{tikzcd}$$

where the morphisms $\widehat{\rho}_a$ and $\widehat{\delta}$ in the diagram are induced by the corresponding morphisms of the diagram (7).

The map

$$\kappa : \mathbb{Z}[PM_n] \to \widehat{\mathcal{PM}}_n$$

can be defined following the same steps as the definition of the universal Vassiliev invariant. However it is more simple to use the universal invariant for the classical pure braid group and define κ as the following composition

$$\mathbb{Z}[PM_{n+1}] \to \mathbb{Z}[P_n] \xrightarrow{\mu} \widehat{\mathcal{P}_n} \to \widehat{\mathcal{PM}}_{n+1},$$

where the first map is the canonical inclusion and the last one is the canonical projection.

We can also reason inversely: at first construct κ, then define the map $\widehat{\kappa \otimes \chi}$ as the composition

$$\mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \xrightarrow{\kappa \otimes \chi} \widehat{\mathcal{PM}_{n+1}} \otimes \mathbb{Z}[[y]] \to \widehat{\mathcal{PM}_{n+1}} \widehat{\otimes} \mathbb{Z}[[y]],$$

where the last map is the completion, and then define μ using the following diagram

$$\begin{array}{ccc} \mathbb{Z}[P_n] & \cong & \mathbb{Z}[PM_{n+1}] \otimes \mathbb{Z}[C] \\ \Big\downarrow \mu & & \Big\downarrow \widehat{\kappa \otimes \chi} \\ \widehat{\mathcal{P}}_n & \cong & \widehat{\mathcal{PM}}_{n+1} \widehat{\otimes} \mathbb{Z}[[y]]. \end{array} \tag{14}$$

The map μ defined by (14) is a universal Vassiliev invariant for the classical braids, though it may not coincide with the map constructed for the classical pure braid group which is not unique.

Theorem

The map

$$\kappa : \mathbb{Z}[PM_n] \to \widehat{\mathcal{PM}}_n$$

respects the filtration and induces a multiplicative isomorphism at the associated graded level.

We define the map

$$\lambda : \mathbb{Z}[P_n(S^2)] \to \widehat{\mathcal{P}_n}(S^2)$$

using the following diagram

$$\begin{tikzcd} \mathbb{Z}[PM_n] \otimes \mathbb{Z}[C_2] \arrow[d, "\widehat{\kappa \otimes \alpha}"] \arrow[r, phantom, "\cong"] & \mathbb{Z}[P_n(S^2)] \arrow[d, "\lambda"] \\ \widehat{\mathcal{PM}_n} \widehat{\otimes} \widehat{\mathcal{A}} \arrow[r, phantom, "\cong"] & \widehat{\mathcal{P}_n}(S^2). \end{tikzcd} \tag{15}$$

Theorem

The map

$$\lambda : \mathbb{Z}[P_n(S^2)] \to \widehat{\mathcal{P}_n}(S^2)$$

respects the filtration, induces a multiplicative isomorphism at the associated graded level and fits in the following diagram of filtered rings

$$\begin{tikzcd} \mathbb{Z}[P_n] \arrow[r, "\rho_p"] \arrow[d, "\mu"] & \mathbb{Z}[P_n(S^2)] \arrow[d, "\lambda"] \\ \widehat{\mathcal{P}}_n \arrow[r, "\widehat{\rho}_a"] & \widehat{\mathcal{P}_n}(S^2). \end{tikzcd} \tag{16}$$

The symmetric group Σ_n acts on the algebras $\widehat{\mathcal{PM}}_n$ and $\widehat{\mathcal{P}_n}(S^2)$ by the action on the indices of $x_{i,j}$:

$$\sigma(x_{i,j}) = x_{\sigma(i),\sigma(j)}, \quad \sigma \in \Sigma_n.$$

This action preserves the defining relations (12) and (10). We define the following filtered algebras as the semi-direct products with respect to the given action:

$$\widehat{\mathcal{M}_n} = \widehat{\mathcal{PM}}_n \rtimes \mathbb{Z}[\Sigma_n], \tag{17}$$

$$\widehat{\mathcal{B}_n}(S^2) = \widehat{\mathcal{P}_n}(S^2) \rtimes \mathbb{Z}[\Sigma_n]. \tag{18}$$

According to the Markov normal form for $B_n(S^2)$ proved by R. Gillet and J. Van Buskirk every element b of $B(S^2)$ can be written uniquely in the form

$$b = qS(p),$$

where $q \in P_n(S^2)$ and p is the permutation defined by the braid b. We define the map

$$K : \mathbb{Z}[B_n(S^2)] \to \widehat{\mathcal{B}_n}(S^2)$$

by the formula

$$K(b) = \lambda(q) \otimes p. \tag{19}$$

The map

$$K_M : \mathbb{Z}[M_n] \to \widehat{\mathcal{M}}_n$$

is defined similarly using κ instead of λ in (19).

Theorem

The homomorphisms of abelian groups

$$K_M : \mathbb{Z}[M_n] \to \widehat{\mathcal{M}}_n,$$

$$K : \mathbb{Z}[B_n(S^2)] \to \widehat{\mathcal{B}_n}(S^2)$$

are injections, they respect the filtration, induce a multiplicative isomorphisms at the associated graded level and fit in the following diagram of filtered rings

$$\begin{tikzcd}
\mathbb{Z}[B_n(S^2)] \arrow[r, "\rho_p"] \arrow[d, "K"] & \mathbb{Z}[M_n] \arrow[d, "K_M"] \\
\widehat{\mathcal{B}}_n(S^2) \arrow[r, "\widehat{\rho_a}"] & \widehat{\mathcal{M}}_n,
\end{tikzcd}$$

which leads to the following diagram with isomorphisms

$$
\begin{tikzcd}
\mathbb{Z}[B_n(S^2)] \cong \mathbb{Z}[M_n] \otimes \mathbb{Z}[C_2] \arrow[d, "K"] & \arrow[d, "\widehat{K_M \otimes \alpha}"] \\
\widehat{\mathcal{B}}_n(S^2) \quad \cong & \widehat{\mathcal{M}}_n \hat{\otimes} \widehat{\mathcal{A}}.
\end{tikzcd}
\tag{20}
$$

Corollary

The groups $\mathbb{Z}[M_n]/\Phi^m(\mathbb{Z}[M_n])$ *and* $\widehat{\mathcal{M}_n}/\Phi^m(\widehat{\mathcal{M}_n})$ *are isomorphic and are torsion free. There are also isomorphisms of abelian groups.*

$$\mathbb{Z}[B_n(S^2)]/\Phi^{m+1}(\mathbb{Z}[B_n(S^2)]) \cong \widehat{\mathcal{B}_n}(S^2)/\Phi^{m+1}(\widehat{\mathcal{B}_n}(S^2)) \cong$$

$$(\widehat{\mathcal{M}_n}/\Phi^{m+1}(\widehat{\mathcal{M}_n})) \oplus (\bigoplus_{i=0}^{m-1} \Phi^i(\widehat{\mathcal{M}_n})/\Phi^{i+1}(\widehat{\mathcal{M}_n}) \otimes \mathbb{Z}/2^{m-i}). \quad (21)$$

Corollary

There exist two elements in $B_n(S^2)$ which are not distinguished by any Vassiliev invariant to an abelian group without 2-torsion. For any couple of elements $a \neq b$ of $B_n(S^2)$ there exists a natural number k and Vassiliev invariant v to an abelian group A with an element of order 2^k such that v distinguish a and b:

$$v(a) \neq v(b).$$

M. Eisermann gave an example of a couple of elements in $B_n(S^2)$ which are not distinguished by Vassiliev invariants to an abelian group without 2-torsion. These elements are: the trivial braid that represents the unit in the braid group $B_n(S^2)$ and the braid $\tau = (\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1})^n$. These elements correspond to element 1 and a in $\mathbb{Z}[PM_n]$. By the formulae (13) a maps to $1 + x \in \widehat{\mathcal{A}}$. The difference between 1 and $1 + x$ is equal to x and according to (21) x corresponds to a generator of $\mathbb{Z}/2^k$ in

$$\mathbb{Z}[B_n(S^2)]/\Phi^{m+1}(\mathbb{Z}[B_n(S^2)]).$$

So it is mapped to 0 by any map to an abelian group without 2-torsion.

Examples

1. $n = 4$. The pure braid group $P_4(S^2)$ of a 2-sphere is isomorphic to the direct product of the cyclic group of order 2 (generated by Δ^2) and the pure braid group on one strand of a 2-sphere with three points deleted, it is the fundamental group of disc with two points deleted, that is a free group F_2 on two generators. Its associated graded Lie algebra is a direct sum of central $\mathbb{Z}/2$ and the free Lie algebra on two generators. The pure mapping class group PM_4 is isomorphic to a free group on two generators. According to Theorem 4 its associated graded Lie algebra is the free Lie algebra on two generators. The universal Vassiliev invariant for PM_4 is nothing but Magnus expansion $\mathbb{Z}[F_2] \xrightarrow{\mu_e} \mathbb{Z}\langle\langle x_1, x_2\rangle\rangle$ and the universal invariant for $P_4(S^2)$ is

$$\mathbb{Z}[F_2] \otimes \mathbb{Z}[C_2] \overset{\widehat{\mu_e \otimes \alpha}}{\longrightarrow} \mathbb{Z}\langle\langle x_1, x_2\rangle\rangle \widehat{\otimes} \widehat{\mathcal{A}}.$$

2. $n = 5$. The pure mapping class group PM_5 is isomorphic to a semi-direct product of a free group on three generators and a free group on two generators.

$$PM_5 \cong F\langle a_{1,2}, a_{1,3}, a_{1,4}\rangle \rtimes F\langle a_{2,3}, a_{2,4}\rangle.$$

We write every element of PM_5 in the Markov normal form

$$f_3 f_2, \quad \text{where} f_3 \in F\langle a_{1,2}, a_{1,3}, a_{1,4}\rangle, \ f_2 \in F\langle a_{2,3}, a_{2,4}\rangle$$

and we define the universal invariant $\mu : PM_5 \to \widehat{\mathcal{M}}_5$ by the formula

$$\mu(f_3 f_2) = \mu_3(f_3)\, \mu_2(f_2),$$

where μ_3 and μ_2 are defined as follows

$$\mu_3(a_{1,j}) = 1 + B_{1,j}, \ j = 2, 3, 4,$$

$$\mu_2(a_{2,k}) = 1 + B_{2,k}, \ k = 3, 4.$$