LOCAL REPRESENTATIONS OF BRAID GROUP AND ITS GENERALIZATIONS

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Recall the braid group B_n, $n \geq 2$, on n strands is defined by generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the defining relations

$$\begin{array}{rcll} \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } \ i = 1, 2, ..., n-2, \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for } \ |i-j| \geq 2. \end{array}$$

The group B_n embeds in the automorphism group $\mathrm{Aut}(F_n)$ of the free group $F_n = < x_1, ..., x_n >$. Here the generator σ_i, $i = 1, 2, ..., n-1$, defines the automorphism $\sigma_i : B_n \to \mathrm{Aut}(F_n)$,

$$\sigma_i : \begin{cases} x_i \to x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \to x_i, \\ x_j \to x_j, j \neq i, i+1. \end{cases}$$

This representation is referred to as *the Artin representation*.

The Burau representation

Using the Magnus approach and Fox derivatives from the Artin representation the Burau one is constructed.

$$\rho_B : B_n \to GL_n(\mathbf{Z}[t^{\pm 1}]).$$

$$\rho_B(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \quad i = 1, 2, ..., n-1.$$

Recall a representation $\varphi : B_n \to GL_n(\mathbb{C})$ is referred to as *local* if

$$\varphi(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & * & * & 0 \\ 0 & * & * & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right) = \left(\begin{array}{c|c|c} I_{i-1} & 0 & 0 \\ \hline 0 & R_i & 0 \\ \hline 0 & 0 & I_{n-i-1} \end{array}\right),$$

$i = 1, 2, ..., n-1$, where I_m is the identity matrix of order m and R_i is a matrix of order 2.

A local representation is referred to as *homogeneous* if $R_1 = R_2 = ... = R_{n-1}$.

Theorem 1. *Provided a local representation $\varphi : B_3 \to GL_3(\mathbb{C})$, φ coincides with one of the the two types of representations:*

$$1)\ \varphi(\sigma_1) = \left(\begin{array}{cc|c} \alpha(1-d) & \frac{(1-d)(1-\alpha+d\alpha)}{c} & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

$$\varphi(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \alpha & \frac{(1-\alpha)(1-d+d\alpha)}{\gamma} \\ 0 & \gamma & d(1-\alpha) \end{array}\right), \textit{ where } d, \alpha \neq 1,\ c, \gamma \neq 0;$$

$$2)\ \varphi(\sigma_1) = \left(\begin{array}{cc|c} 0 & b & 0 \\ c & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),\ \varphi(\sigma_2) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & \frac{bc}{\gamma} \\ 0 & \gamma & 0 \end{array}\right), \textit{ where } bc, \gamma \neq 0.$$

Theorem 2. *Given* $\varphi : B_n \to GL_n(\mathbb{C})$ *a local homogeneous representation,* φ *coincides with one of the representations* $\varphi_1, \varphi_2, \varphi_3$ *defined as follows:*

$$\varphi_j : B_n \to GL_n(\mathbb{C}).$$

$$1)\, \varphi_1(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & \alpha & \frac{1-\alpha}{\gamma} & 0 \\ 0 & \gamma & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ \gamma \neq 0, \ i = 1, 2, ..., n-1;$$

$$2)\, \varphi_2(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1-d}{c} & 0 \\ 0 & \mathrm{c} & \mathrm{d} & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ c \neq 0, \ i = 1, 2, ..., n-1;$$

$$3)\, \varphi_3(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{b} & 0 \\ 0 & \mathrm{c} & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ bc \neq 0, \ i = 1, 2, ..., n-1.$$

Note if $\gamma = 1$, $\alpha = 1 - t$ *in* φ_1 *we obtain the Burau representation.*

Wada Representations

Wada constructed four new local representations of B_n in $\mathrm{Aut}(F_n)$ defined as follows:

$$w_1^{(k)}(\sigma_i) : \begin{cases} x_i \to x_i^k x_{i+1} x_i^{-k}, \\ x_{i+1} \to x_i, \\ x_j \to x_j, j \neq i, i+1. \end{cases}$$

Note given $k = 1$, this is the Artin representation.

$$w_2(\sigma_i) : \begin{cases} x_i \to x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \to x_i, \\ x_j \to x_j, j \neq i, i+1. \end{cases}$$

$$w_3(\sigma_i) : \begin{cases} x_i \to x_i x_{i+1} x_i, \\ x_{i+1} \to x_i^{-1}, \\ x_j \to x_j, j \neq i, i+1. \end{cases}$$

$$w_4(\sigma_i) : \begin{cases} x_i \to x_i^2 x_{i+1}, \\ x_{i+1} \to x_{i+1}^{-1} x_i^{-1} x_{i+1}, \\ x_j \to x_j, j \neq i, i+1. \end{cases}$$

Linear representations corresponding to Wada representations

Analogously as the Burau representation is constructed from the Artin's one we use the Magnus approach to construct linear representations corresponding to Wada representations.

$$w_1^{(k)}, w_2, w_3, w_4 : B_n \to \mathrm{Aut}(F_n)$$

$$\rho_1^{(k)}, \rho_2, \rho_3, \rho_4 : B_n \to GL_n(\mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_n^{\pm 1}])$$

Obtained representations are as follows:

$$\rho_1^{(k)}(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 1-t^k & t^k & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \quad i = 1, 2, ..., n-1.$$

Note given $q = t^k$, we obtain the Burau representation.

Linear representations corresponding to Wada representations

$$\rho_2(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ i = 1, 2, ..., n-1.$$

$$\rho_3(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 2 & t_i & 0 \\ 0 & -t_i^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ i = 1, 2, ..., n-1.$$

$$\rho_4(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \\ 0 & \text{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ i = 1, 2, ..., n-1.$$

Using the Theorem 1 result there was constructed the extensions of the local representation of B_3 to B_n.
The most interesting case looks as follows

$$\varphi(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & \alpha & \frac{1-\alpha}{\gamma_i} & 0 \\ 0 & \gamma_i & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right), \ \gamma_i \neq 0, \ i = 1, 2, ..., n-1.$$

Linear local representations of B_n

Theorem. *All linear local representations of B_n are equivalent to the Burau one in some sense. In particular all linear local homogeneous representations of B_n are equivalent to the Burau representations.*

So there does not exist a faithful local representation of B_n to $GL_n(\mathbb{C})$, $n \geq 5$.

Recently the generalizations of braid group are studied such as the virtual braid group VB_n, the welded braid group WB_n and the singular braid group SB_n.

There exist the extensions of the Burau representation to these generalizations.

Using the Theorem 2 results there were constructed another linear local representations of the virtual braid group VB_n, the welded braid group WB_n and the singular braid group SB_n.

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