Bowling ball representation of virtual string links

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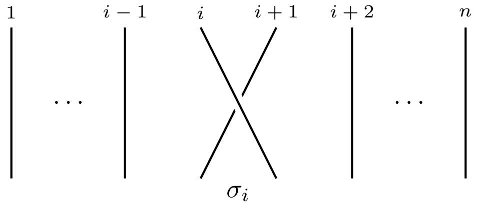
Braids and Burau Representation

Several definitions of B_n

- $B_n = \pi_1(\mathcal{C}_n(\mathbb{C}), *)$, where $\mathcal{C}_n(\mathbb{C}) = \{(z_1, \cdots, z_n) | z_i \in \mathbb{C}, z_i \neq z_j \forall i \neq j\}/\Sigma_n$.
- $B_n = \mathcal{M}_{0,1,n} = \pi_0(\mathrm{Diff}^+(S_{0,1,n}))$, where $S_{0,1,n}$ denotes a disk with n punctures.
- $B_n = \{\sigma_1, \cdots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$.



Some basic properties of B_n

- The center of B_n is nontrivial, which is an infinite cyclic group generated by the full twist $(\sigma_1 \cdots \sigma_{n-1})^n$.
- (Bigelow 2001, Krammer 2002) B_n is a linear group. More precisely the Lawrence – Krammer representation

$$B_n \to GL_{\frac{n(n-1)}{2}}(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$$

  is a faithful representation.
- (Dehornoy 1994) B_n has a right-invariant ordering, i.e. there exists a strict ordering $<$ with the property that if $f < g$ then $fh < gh$. In particular, B_n is torsion free.

The (unreduced) Burau representation $B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$ can be described by mapping

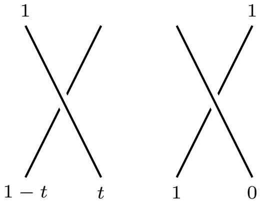
$$\sigma_i \to \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

- The Burau representation is faithful for $n \leq 3$, but non-faithful for $n \geq 5$ (J. A. Moody 1991, D. D. Long, M. Paton 1993, Bigelow 1999).
- (Open problem) Is the Burau representation faithful for $n = 4$?

In his seminal paper [1], Jones mentioned a probabilistic interpretation of the unreduced Burau representation of positive braids.

Interpret a positive braid B_n^+ as a bowling alley with n intertwining lanes. Let us throw a bowling ball down one of the lanes such that when it meets a crossing point it falls down with probability $1 - t$. Then the $(i, j)-$entry of the unreduced Burau representation is the probability that a ball begins at the $i-$th lane and ends up in the $j-$th lane.

[1] V. Jones. Hecke algebra representations of braid groups and link polynomials. Ann. Math., 126 (1987), 335-388



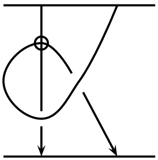
$$\sigma_1 \to \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

- This interpretation of the unreduced Burau representation can be generalized for all B_n. One just need to assume when the ball meets a negative crossing it falls down with probability $1 - t^{-1}$. However this is not a "realistic" probability model.
- Recently[2], Bigelow extended this representation of B_n^+ by allowing multiple bowling balls to be bowled simultaneously.

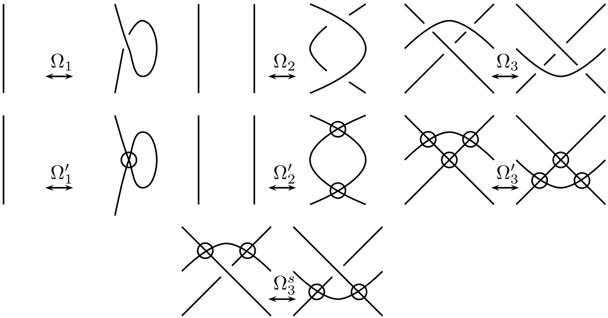
[2] S. Bigelow. Bowling ball representations of braid groups. arXiv:1409.4074v1

Bowling Ball Representation of Virtual String Links

A virtual n-string link diagram is a collection of n immersed strings in the strip $\mathbb{R} \times [0,1]$ such that the i-th string gives an oriented path from $(i,1)$ to $(\pi(i),0)$, here π denotes a permutation of $\{1,\cdots,n\}$. Each crossing of this diagram is either real or virtual.



A virtual n-string link is an equivalence class of virtual n-string link diagrams under generalized Reidemeister moves.

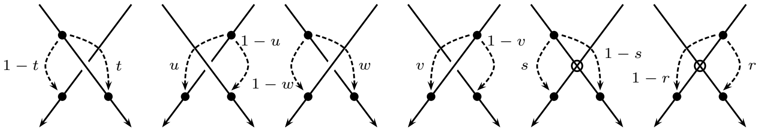


- The set of all virtual n-string links has a monoid structure.
- Note that if each strand meets $\mathbb{R} \times t$ $(0 < t < 1)$ transversely at one point then we obtain the virtual braid VB_n.

Question

Can we find some probability interpretations (maybe unrealistic) of virtual string links?

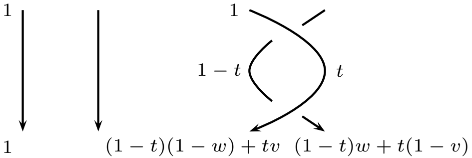
As before let us put a bowling ball at $(i, 1)$ ($i \in \{1, \cdots, n\}$), then we assume this bowling ball will travel along the lane according to the orientation and behave according to the following rules:



According to the definition of virtual string links, we want to ensure that the interpretation does not depend on the choice of the diagram.

For example, the following Ω_2 implies that

$$\begin{cases} (1-t)(1-w)+tv=1 \\ (1-t)w+t(1-v)=0. \end{cases}$$



After checking all generalized Reidemeister moves, we conclude two choices of the bowling ball models:

1. $u = w = 1$, $s = r = 0$ and $v = t^{-1}$,
2. $u = w = t = v = 1$, $r = -s$ and $s^2 = 0$.

Note that the first case is exactly the Burau representation. By defining the (i, j)-th entry to be the "possibility" that a ball descends from $(i, 1)$ to $(j, 0)$, we can associate a matrix $M(L) \in GL_n(\mathbb{Z}[t^{\pm 1}])$ for a given virtual n-string link L, which generates the unreduced Burau representation.

Theorem (X. Lin, F. Tian and Z. Wang 1998)

Each entry of $M(L)$ converges to a rational function of t, and the matrix $M(L)$ is invariant under the generalized Reidemeister moves.

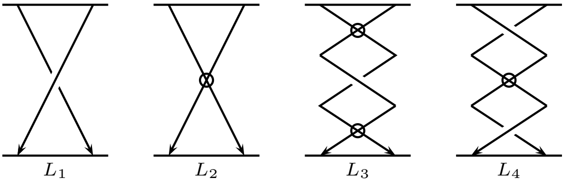
In the remainder of this talk, we will focus on the second case, i.e. $u = w = t = v = 1$, $r = -s$ and $s^2 = 0$. Similarly we can define a matrix $M(L) \in GL_n(\mathbb{Z}[s]/(s^2))$ for each virtual n-string link L.

Theorem (Cheng 2015)

Let L be a virtual n-string link diagram, then we can assign an $n \times n$ matrix $M(L)$ to L such that

1. *Each entry of $M(L)$ has the form $as + b$, here $a \in \mathbb{Z}$ and $b \in \{0, 1\}$;*
2. *$M(L)$ is invariant under generalized Reidemeister moves. Moreover $M(L)$ determines a representation of the monoid of virtual n-string links.*

Some examples



$$M(L_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M(L_2) = \begin{pmatrix} s & 1-s \\ 1+s & -s \end{pmatrix},$$

$$M(L_3) = \begin{pmatrix} 2s & 1-2s \\ 1+2s & -2s \end{pmatrix}, M(L_4) = \begin{pmatrix} -s & 1+s \\ 1-s & s \end{pmatrix}.$$

Hence L_1, L_2, L_3, L_4 are mutually different.

Recall that VB_n, the n-strand virtual braid group is generated by $\sigma_1, \cdots, \sigma_{n-1}$ and $\tau_1, \cdots, \tau_{n-1}$ with relations

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$, if $|i - j| > 1$;
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$;
3. $\tau_i^2 = 1$;
4. $\tau_i \tau_j = \tau_j \tau_i$, if $|i - j| > 1$;
5. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$;
6. $\sigma_i \tau_j = \tau_j \sigma_i$, if $|i - j| > 1$;
7. $\sigma_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \sigma_{i+1}$.

Define a homomorphism $\rho : VB_n \to \mathrm{GL}_n(\mathbb{Z}[s]/(s^2))$ as follows

$$\sigma_i \to I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

$$\tau_i \to I_{i-1} \oplus \begin{pmatrix} s & 1+s \\ 1-s & -s \end{pmatrix} \oplus I_{n-i-1}.$$

Corollary

If $L \in VB_n$, then $M(L)^T = \rho(L)$.

Virtual Flat Biquandle

Now we want to discuss the algebraic structure behind this probabilistic interpretation.

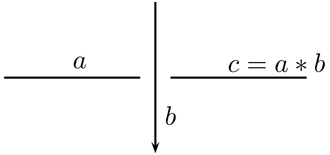
A quandle $(Q, *)$, is a set Q with a binary operation $(a, b) \to a * b$ satisfying the following axioms

1. $\forall a \in Q$, $a * a = a$.
2. $\forall b, c \in Q$, $\exists! a \in Q$ such that $a * b = c$.
3. $\forall a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

Some examples

- For any set Q, define $a * b = a$ for any $a, b \in Q$;
- Let $R_n = \{0, 1, \cdots, n-1\}$, define $i * j = 2j - i \pmod{n}$;
- On S^2, define $x * y = 2(x \cdot y)y - x$ for any $x, y \in S^2$.

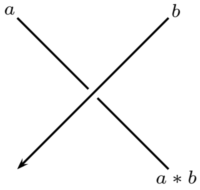
Given a quandle Q and a knot diagram K, assign each arc of K with an element of Q such that at each crossing the following relation is satisfied.



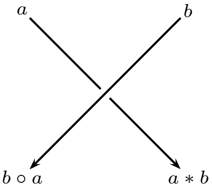
Theorem

The number of colorings $Col_Q(K)$ is a knot invariant.

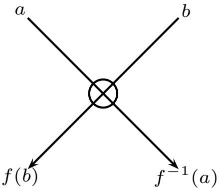
- Quandle was introduced by Joyce (1982) and Matveev (1984) independently.
- Fenn, Jordan-Santana and Kauffman introduced the notion of biquandle in 2004.
- The virtual biquandle was proposed by Kauffman and Manturov in 2005.



quandle



biquandle



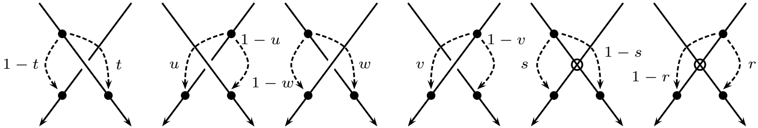
virtual biquandle

In 2012 Kauffman considered the notion of flat biquandle, which was named as semiquandle by Henrich and Nelson (2010).
By a flat biquandle, we mean a set FB with two binary operations denote $a * b$ and $a \circ b$ satisfying the following axioms:

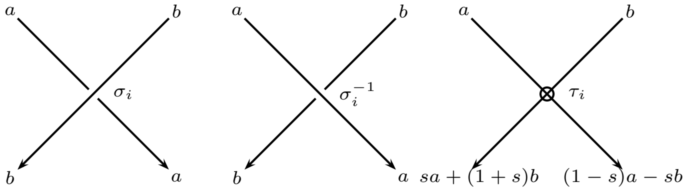
1. $\forall a \in FB, \exists! x, y \in FB$ such that
   $a \circ x = x, x * a = a, y \circ a = a, a * y = y$;
2. $\forall a, b \in FB, \exists! x, y \in FB$ such that
   $x = b \circ y, y = a \circ x, b = x * a, a = y * b$, and
   $(a \circ b) * (b * a) = a, (b * a) \circ (a \circ b) = b$;
3. $\forall a, b, c \in FB$, we have
   $(a \circ b) \circ c = (a \circ (c * b)) \circ (b \circ c), (c * b) * a =$
   $(c * (a \circ b)) * (b * a), (b \circ c) * (a \circ (c * b)) = (b * a) \circ (c * (a \circ b))$.

Recall our assumption of the probability model,

$$u = w = t = v = 1,\ r = -s \text{ and } s^2 = 0.$$



Under this assumption, we have



Definition

A virtual flat biquandle is a set VFB with two binary operations denoted by $a * b$ and $a \circ b$. If we denote $a * b$ and $a \circ b$ by $S_b(a)$ and $T_b(a)$ respectively, then $S_a, T_a : VFB \to VFB$ satisfy the following axioms:

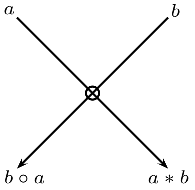
1. $S_a S_b = S_b S_a, T_a T_b = T_b T_a, S_a T_b = T_b S_a$;
2. $S_a = S_{T_b(a)} = S_{S_b(a)}, T_a = T_{S_b(a)} = T_{T_b(a)}$;
3. $T_a S_a = S_a T_a = id$.

Let $S = \mathbb{Z}[s]/(s^2)$ with two binary operations

$$S_a(b) = -sa + (1-s)b \text{ and } T_a(b) = sa + (1+s)b,$$

then S is the virtual flat biquandle which we used in the probability interpretation of virtual string links.

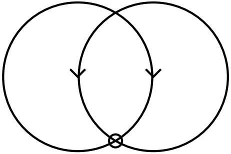
Let L be a flat virtual link diagram, the fundamental virtual flat biquandle $VFB(L)$ is generated by the v-arcs (a part of the diagram from a virtual crossing to the next virtual crossing) of the diagram under the equivalence relation generated by the virtual flat biquandle axioms and the relations at virtual crossings.



Theorem (Cheng 2015)

$VFB(L)$ is a flat virtual link invariant. In particular the cardinality of the set of virtual flat biquandle homomorphisms from $VFB(L)$ to S is an invariant of L, here S denotes a given virtual flat biquandle.

As an example, let H and T denote the flat virtual Hopf link and 2-component trivial link respectively.



- $S_1 = \{x, y | x * x = x \circ x = x * y = x \circ y = x, y * x = y \circ x = y * y = y \circ y = y\}$, then $\mathrm{Col}_{S_1}(H) = 4 = \mathrm{Col}_{S_1}(T)$.
- $S_2 = \{x, y | x * x = x \circ x = x * y = x \circ y = y, y * x = y \circ x = y * y = y \circ y = x\}$, then $\mathrm{Col}_{S_1}(H) = 0 \neq 4 = \mathrm{Col}_{S_1}(T)$.

Question

Can we use S_1 to distinguish between H and T?

Cocycle Invariants

- The homology theory of rack was proposed by R. Fenn, C. Rourke, B. Sanderson in 1995.
- The homology theory of quandle was introduced by J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito in 2003.
- The homology theory of virtual biquandle was defined by Ceniceros and Nelson in 2009.

In general a 2-cocycle of rack/quandle/virtual biquandle can offer an extended invariant of the associated coloring invariant.

Given a virtual flat biquandle S, let $C_n(S)$ denote the free abelian group generated by n-tuples $(a_1, \cdots, a_n)$. Consider the boundary map $\partial_n : C_n(S) \to C_{n-1}(S)$ defined by

$$\partial_n(a_1, \cdots, a_n) = \sum_{i=1}^{n} (-1)^i ((a_1 * a_i, \cdots, a_{i-1} * a_i, a_{i+1}, \cdots, a_n) - (a_1, \cdots, a_{i-1}, a_{i+1} \circ a_i, \cdots, a_n \circ a_i))$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$.

Lemma

$\partial_{n-1}\partial_n = 0.$

Therefore $C_*(S) = \{C_n(S), \partial_n\}$ is a chain complex. Let $C'_n(S)$ be a subset of $C_n(S)$ generated by

$$(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) + (a_1, \cdots, a_{i+1} \circ a_i, a_i * a_{i+1}, \cdots, a_n)$$

for $n \geq 2$, and $C'_n(S) = 0$ for $n \leq 1$.

Lemma

$C'_*(S) = \{C'_n(S), \partial_n\}$ *is a sub-complex of* $C_*(S)$.

Let $C_*^{VF}(S)$ be the quotient complex $C_*(S)/C'_*(S)$ and A an abelian group without 2-torsion, then we consider the homology and cohomology groups

$$H_n^{VF}(S; A) = H_n(C_*^{VF}(S) \otimes A), H^n_{VF}(S; A) = H^n(\mathrm{Hom}(C_*^{VF}(S), A)).$$

We define another boundary map $d_n : C_n(S) \to C_{n-1}(S)$ as below

$$d_n(a_1, \cdots, a_n) = \sum_{i=1}^{n-1} (-1)^i ((a_1, \cdots, \widehat{a_i}, \cdots, a_n) - (a_1, \cdots, \widehat{a_i}, \cdots, a_{n-1}, a_n \circ a_i)).$$

Lemma

$d_{n-1} d_n = 0.$

It follows that $C_*^{SF}(S) = \{C_n(S), d_n\}$ is a chain complex and we can define the the homology and cohomology groups

$$H_n^{SF}(S; A) = H_n(C_*^{SF}(S) \otimes A), H^n_{SF}(S; A) = H^n(\mathrm{Hom}(C_*^{SF}(S), A)).$$

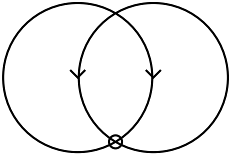
Let L be a flat virtual link, and S a finite virtual flat biquandle. Assume $\phi : S \times S \to A$ represents a 2-cocycle in both $C^2_{VF}(S; A)$ and $C^2_{SF}(S; A)$, then for a fixed coloring $\theta : VFB(L) \to S$ we can assign a (Boltzmann) weight to each virtual crossing of L. Then the state-sum $\Phi_\phi(L)$, which takes value in the group ring $\mathbb{Z}[A]$, has the following expression

$$\Phi_\phi(L) = \sum_\theta \prod_\tau B(\tau, \theta),$$

here the product is taken over all virtual crossings and the sum is taken over all colorings.

Theorem (Cheng 2015)

$\Phi_\phi(L)$ *is an invariant of* L*.*



Recall that the coloring invariant $\mathrm{Col}_{S_1}(H)$ $=\mathrm{Col}_{S_1}(T) = 4$. Here S_1 is the virtual flat biquandle $\{x, y | x * x = x \circ x = x * y = x \circ y = x, y * x = y \circ x = y * y = y \circ y = y\}$.

Let us consider the map $\phi : S_1 \times S_1 \to \mathbb{Z}$ defined by

$$\phi(x, y) = -\phi(y, x) = 1 \text{ and } \phi(x, x) = \phi(y, y) = 0.$$

It satisfies the 2-cocycle condition of $C^2_{VF}(S; A)$ and $C^2_{SF}(S; A)$. Direct calculation shows that

$$\Phi_\phi(H) = 1 + (-1) + 0 + 0 \text{ but } \Phi_\phi(T) = 0 + 0 + 0 + 0.$$

Thank you!