Representations of the virtual braid groups to the rook algebras

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Definition

Braid group on n strands, denoted by $\mathbb{B}_n$, is group generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1$$
$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \quad \text{if } i = 1, \ldots, n-1$$

Definition

Virtual braid group on n strands, denoted by $V\mathbb{B}_n$, is group with generators:

$$\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}$$

and relations:

$$\begin{gathered}
\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \\
\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \quad \text{if } i = 1, \dots, n-1 \\
\rho_i^2 = e \quad \text{if } i = 1, \dots, n-1 \\
\rho_{i+1}\rho_i\rho_{i+1} = \rho_i\rho_{i+1}\rho_i \quad \text{if } i = 1, \dots, n-1 \\
\rho_i\rho_j = \rho_j\rho_i \quad \text{if } |i-j| > 1 \\
\rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1} \quad \text{if } i = 1, \dots, n-1 \\
\sigma_i\rho_j = \rho_j\sigma_i \quad \text{if } |i-j| > 1
\end{gathered}$$

Definition

Let R_n denote set of n $\times$ n matrices with entries from $\{0, 1\}$ having at most one 1 in each row and in each column.

Example for $n = 2$

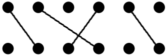
$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Proposition

R_n with standard matrix multiplication is monoid.

Definition

Rook diagram – bipartite graph with n vertices in each partite set, where one belong of the top and another on bottom of a rectangle, such that each vertex has degree either zero or one.

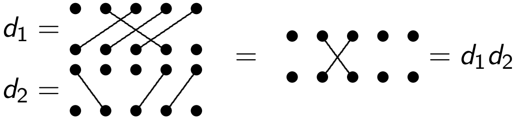


$$\leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There is one-to-one correspondence between rook diagrams and matrices of R_n

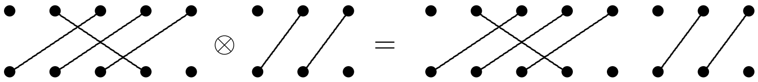
Definition

Let d_1 and d_2 be rook diagrams with the same number $2n$ of vertices. The product, $d_1 d_2$, is rook diagrams with $2n$ vertices and edges, defined of the rule, represented of the following picture.



Definition

Given diagrams d_1 and d_2, we define the tensor product, denoted $d_1 \otimes d_2$, to be the result of appending of d_2 to the right of d_1.



Definition

Diagram from R_n is planar if it can be drawn (keeping inside of the rectangle formed by its vertices) without any crossings of edges.
Let P_n – set of planar diagrams of R_n.

Proposition

P_n is a monoid, called planar rook monoid.

Definition

$\mathbb{C}R_n$ – rook algebra – $\mathbb{C}$-algebra generated by R_n with multiplication defined using the distributive law and multiplication in R_n.

Definition

$\mathbb{C}P_n$ – planar rook algebra – $\mathbb{C}$-algebra generated by P_n with multiplication defined using the distributive law and multiplication in P_n.

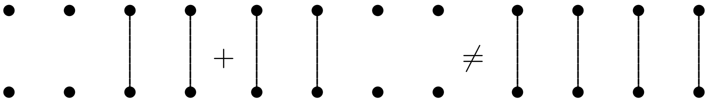
The following inequivalence holds in $\mathbb{C}R_n$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

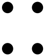
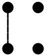
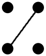
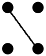
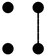
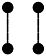
The following inequivalence holds in $\mathbb{C}R_n$:

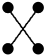
$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\neq\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

But this is more obvious if we are considering the diagrams.



We will denote the elements of R_2 in following fashion:

$d_1 =$  $d_2 =$  $d_3 =$  $d_4 =$  $d_5 =$  $d_6 =$ 

$d_7 =$ 

Define mapping $\varphi : \mathbb{B}_n \to \mathbb{C}P_n$ by the following rule:

$$\varphi(\sigma_i) = a \cdot d_{1i} + b \cdot d_{2i} + c \cdot d_{3i} + d \cdot d_{4i} + e \cdot d_{5i} + \cdot d_{6i}$$

where $a, b, c, d, e \in \mathbb{C}$ and I is the identity diagram in P_1

$$d_{ji} = I^{\otimes i-1} \otimes d_j \otimes I^{\otimes n-i-1}$$

Question

For which coefficients φ is homomorphism?

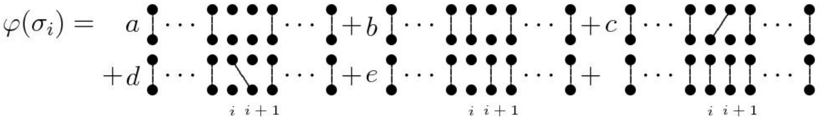
Theorem (S.Bigelow, E.Ramos, R. Yi)

Assuming $a + c + d \neq 1$ and $cd \neq 0$, any mapping of the above form is a homomorphism if and only if its coefficients are in one of the following families:

1. $b = e = -1$
2. $a = -c - d$, $b = -1$, $e = -cd$
3. $a = 1 - c - d + cd$, $b = -1$, $e = -cd$

Basic properties of this representation are:

1. for any braid β, $\varphi(\beta)$ is planar
2. φ is local



Question

Can we extend this representation for virtual braid groups, retaining this properties?

Theorem

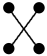
There is no representation $\psi : V\mathbb{B}_n \to \mathbb{C}P_n$, satisfying the following conditions:

1. ψ is local
2. Restriction of ψ on $\mathbb{B}_n$ is φ

Definition

Define mapping $\psi : V\mathbb{B}_n \to \mathbb{C}R_n$ by the next rule:

$$\psi(\sigma_i) = \varphi(\sigma_i)$$
$$\psi(\rho_i) = d_{i,7}$$

$d_7 =$ 

Theorem

The mapping ψ is representation of $V\mathbb{B}_n$.

Theorem (T.Kadokami)

Let $b_1, b_2 \in V\mathbb{B}_2$ – two braids, then b_1 and b_2 correspond to the same link $\Leftrightarrow$ there is $a \in V\mathbb{B}_2$, such that $b_1 = ab_2a^{-1}$.

For any $\beta \in V\mathbb{B}_2$ define linear function $\mathsf{L}_\beta : \mathbb{C}R_2 \to \mathbb{C}R_2$, acting under the following rule:

$$\mathsf{L}_\beta(x) = \varphi(\beta) \cdot x, \forall x \in \mathbb{C}R_2$$

Let $\mathsf{T}(\beta)$ be trace of L_β as trace of linear function.

Theorem

$\mathsf{T}(\beta)$ – *invariant of the link, corresponding braid* β, $\forall \beta \in V\mathbb{B}_2$

Theorem

T *satisfies following skein relations:*
for $a = 1 - c - d + cd, b = -1, e = -cd,$

$$\mathsf{T}(\sigma^2) = (1 - cd) \cdot \mathsf{T}(\sigma) + cd \cdot \mathsf{T}(e)$$

for $a = -c - d, b = -1, e = -cd,$

$$\mathsf{T}(\sigma^2) = (\frac{1}{\sqrt{cd}} - \sqrt{cd}) \cdot \mathsf{T}(\sigma) + \mathsf{T}(e)$$

Definition

For any $t \in \mathbb{C}$ linear function $\partial : \mathbb{C}R_2 \to \mathbb{C}R_2$ defined by the rule:

$$\partial(d_1) = \partial(d_6) = \partial(d_7) = 0$$
$$\partial(d_2) = t(d_3 - d_4) = -\partial(d_5)$$
$$\partial(d_3) = t(d_2 - d_5) = -\partial(d_4)$$

Lemma

Function ∂ is a derivation on $\mathbb{C}R_2$, it satisfy following property

$$\partial(D_1 D_2) = \partial(D_1)D_2 + D_1\partial(D_2),\ \forall D_1, D_2 \in \mathbb{C}R_2$$

For any $\beta \in V\mathbb{B}_2$ and $k \in \mathbb{N}$ define linear function $\mathsf{L}_{k,\beta} : \mathbb{C}R_2 \to \mathbb{C}R_2$, acting according to the rule:

$$\mathsf{L}_{k,\beta}(x) = (\partial^k \varphi(\beta)) \cdot x, \forall x \in \mathbb{C}R_2$$

Let $\mathsf{T} \circ \partial^{\mathsf{k}}(\beta)$ be the trace of $\mathsf{L}_{k,\beta}$ as trace linear function.

Theorem

$\mathsf{T} \circ \partial^{\mathsf{k}}(\beta)$ – invariant of the link, corresponding braid β, $\forall \beta \in V\mathbb{B}_2$

Example

$$\mathsf{T}(\sigma^2\rho\sigma\rho\sigma^2\rho\sigma) = \mathsf{T}(\sigma^2\rho\sigma^2\rho\sigma\rho\sigma)$$

$$\mathsf{T} \circ \partial(\sigma^2\rho\sigma\rho\sigma^2\rho\sigma) \neq \mathsf{T} \circ \partial(\sigma^2\rho\sigma^2\rho\sigma\rho\sigma)$$

Thank you for attention!