On the skein polynomials for knots and links

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Second China-Russia Workshop on
Knot Theory and Related Topics

Sobolev Institute of Mathematics, Novosibirsk
August 21–25, 2015

Abstract

We report new characterizations of the skein (or HOMFLY) polynomial and the Jones polynomial. The usual 'smoothing of crossings' move is not used.

As by-products we obtain characterizations of HOMFLY and Jones polynomials in the realm of knots, where the 'smoothing of crossings' move is not applicable.

The method of proof is similar to our previous work on the Conway potential function.

Outline

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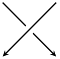
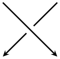
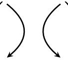
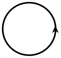
1. The skein and Jones polynomials
2. New characterizations
3. Characterizations for knots

The skein and Jones polynomials

The skein (or HOMFLY) polynomial $P_L(a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$, is an invariant for oriented links. Here $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is the ring of Laurent polynomials in two variables a and z, with integer coefficients. It is defined to be the invariant of oriented links satisfying the axioms

(I) $a^{-1} \cdot P\Big($  $\Big) - a \cdot P\Big($  $\Big) = z \cdot P\Big($  $\Big)$;

(O) $P\Big($  $\Big) = 1.$

The Alexander-Conway polynomial $\Delta_L \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ and the Jones polynomial $V_L \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ are related to the skein polynomial:

$$\Delta_L(t) = P_L(1, t^{\frac{1}{2}} - t^{-\frac{1}{2}}), \qquad V_L(t) = P_L(t, t^{\frac{1}{2}} - t^{-\frac{1}{2}}).$$

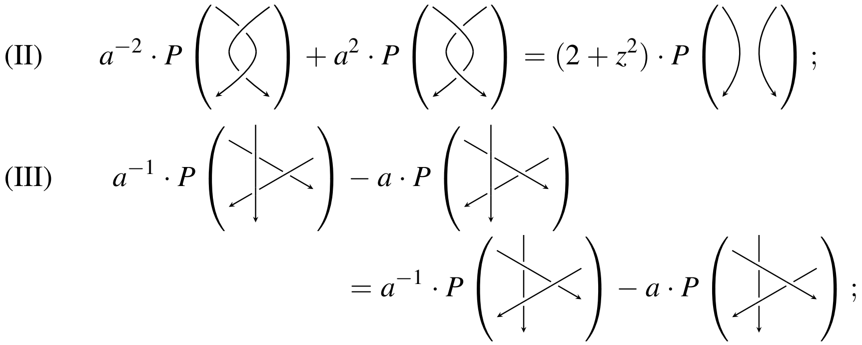
Outline

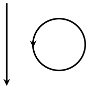
New characterization for P_L

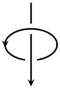
Our main result is

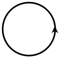
Theorem

The skein polynomial $P_L \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is the invariant of oriented links determined uniquely by the following five axioms.



(IO) $P\Bigg($  $\Bigg) = z^{-1}(a^{-1} - a) \cdot P\Bigg($  $\Bigg)$;

(Φ) $P\Bigg($  $\Bigg) = az^{-1}(1 + z^2 - a^2) \cdot P\Bigg($  $\Bigg)$;

(O) $P\Bigg($  $\Bigg) = 1.$

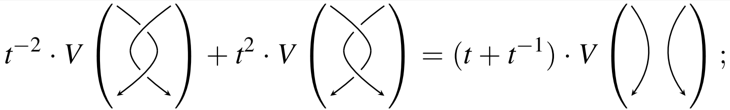
New characterization for V_L

A parallel result is for Jones polynomial. It is not a direct corollary of the above theorem, because the substitutions $a \mapsto t$ and $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ do not send $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ into $Z[t^{\pm\frac{1}{2}}]$.

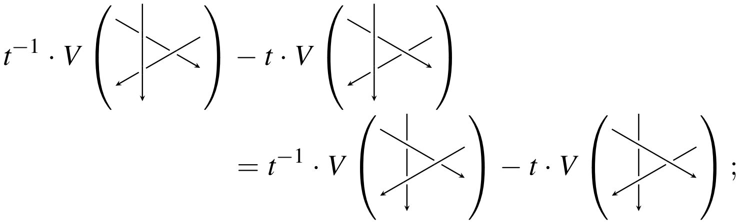
Theorem

The Jones polynomial $V_L \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is the invariant of oriented links determined uniquely by the following five axioms.

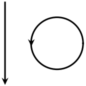
(II$_{\mathrm{J}}$)

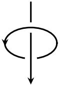


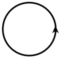
(III$_{\mathrm{J}}$)



New characterization for V_L (cont'd)

$(\mathrm{IO_J})$ $V\Bigg($  $\Bigg) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot V\Bigg($  $\Bigg);$

(Φ_J) $V\Bigg($  $\Bigg) = -t^{\frac{3}{2}}(t + t^{-1}) \cdot V\Bigg($  $\Bigg);$

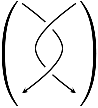
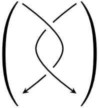
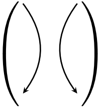
$(\mathrm{O_J})$ $V\Bigg($  $\Bigg) = 1.$

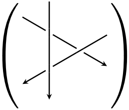
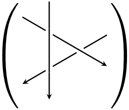
Outline

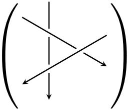
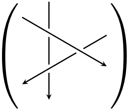
Characterization of P_K for knots

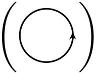
Corollary

The skein polynomial P_K is the invariant of oriented knots determined uniquely by the following three axioms.

(II) $a^{-2} \cdot P$  $+ a^2 \cdot P$  $= (2 + z^2) \cdot P$  ;

(III) $a^{-1} \cdot P$  $- a \cdot P$ 

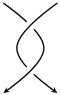
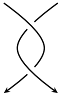
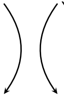
$= a^{-1} \cdot P$  $- a \cdot P$  ;

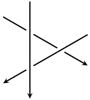
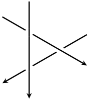
(O) P  $= 1.$

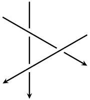
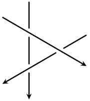
Characterization of V_K for knots

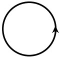
Corollary

The Jones polynomial V_K is the invariant of oriented knots determined uniquely by the following three axioms.

(II$_\mathrm{J}$) $t^{-2} \cdot V$ () $+\, t^{2} \cdot V$ () $= (t + t^{-1}) \cdot V$ () ;

(III$_\mathrm{J}$) $t^{-1} \cdot V$ () $-\, t \cdot V$ ()

$= t^{-1} \cdot V$ () $-\, t \cdot V$ () ;

(O$_\mathrm{J}$) V () $= 1.$