Maximal arithmetic hyperbolic lattices with fixed invariant trace field

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Outline

1. **Distribution of primes**
2. **Quaternion algebra**
    - Number fields
    - Quaternion algebra
    - Orders in quaternion algebra
3. **Arithmetic hyperbolic 3-manifolds**
    - Arithmetic hyperbolic 3-manifolds
    - Examples
4. **Maximal arithmetic hyperbolic 3-manifolds**
    - Higher dim
    - Volume formula and finiteness of Arith hyper 3-mfd
    - Distribution of Max lattices with a fixed trace field K
    - The proof of the distribution

Let

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The Riemann Hypothesis: Riemann 1859

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\mathfrak{p} \textit{ prime}} \frac{1}{1-\mathfrak{p}^{-s}}$, all zeros of $\zeta(s)$ have real part $\frac{1}{2}$ except trivial ones.

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The prime number theorem: J. Hadamard, independently, C. Vallée-Poussin, 1896

$\pi(x) = \frac{x}{\log x} + o(\frac{x}{\log x})$

Ingham' Theorem, 1932.

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The Riemann Hypothesis: equivalent form

$\pi(x) = \frac{x}{\log x} + o(x^{\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$.

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- Kahn-Markovic: surface subgroups in π_1 of a hyperbolic 3-manifold (Hamenstädt; the existence of surface subgroups of a discrete subgroup of some Lie groups).

Number fields

Let K be a number field, i.e., $K = \frac{\mathbb{Q}[x]}{(f(x))}$, $f(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.
We assume $f(x)$ has n roots, a_1, $a_2 = \bar{a}_1 \in \mathbb{C} - \mathbb{R}$, $a_i \in \mathbb{R}$ for $3 \leq i \leq n$.

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$K \to K_i \hookrightarrow \mathbb{R}$, $3 \leq i \leq n$, are $n - 2$ real places of K.

Quaternion algebra

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Ref: M-F Vigneras, The Arithmetic of Quaternion Algebra.

Example

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There are exactly two quaternions over $\mathbb{R}$, the matrix algebra $M_2(\mathbb{R})$ and the Hamilton quaternion $\mathcal{H}$.

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$A = A_K$ is a quaternion over a number field K, for $3 \le i \le n$,

Definition

$$A \otimes_{K_i} \mathbb{R} = \begin{cases} M_2(\mathbb{R}), & K_i \text{ is a splitting place of } A \\ \text{Hamilton quaternion } \mathcal{H}, & K_i \text{ is a ramified place} \end{cases}$$

Now $\mathfrak{p} \in \mathcal{P}_K$ be a prime ideal, from the valuation by $\mathfrak{p}$, we have a $\mathfrak{p}$-adic field $K_\mathfrak{p}$, and an embedding $K \hookrightarrow K_\mathfrak{p}$.

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Definition

$$Ram(A) = Ram_\infty(A) \cup Ram_f(A),$$
$$Ram_\infty(A) = \{ramified\ real\ places\},$$
$$Ram_f(A) = \{ramified\ finite\ places\ (prime\ ideals)\}$$

Classificition of Quaternion Algebra

Theorem: Maclachlan-Reid's Book

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- For a quaternion algebra A over a number field K, the set $Ram(A)$ is finite and has even cardinality.
- $A_1 = A_2$ iff $Ram(A_1) = Ram(A_2)$
- Conversely, for any S be a finite and even cardinality set of place of K, there is a quaternion algebra A over K with $Ram(A) = S$.

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$\mathcal{O}_2 = \mathcal{O}_1 + \mathbb{Z}\alpha$, $\alpha = \frac{1+i+j+ij}{2}$ is a maximal order.

$x = a \cdot 1 + b \cdot i + c \cdot j + d \cdot ij \in A$, $a, b, c, d \in K$, the conjugate
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Definition

$\mathcal{O}$ be an order, $\mathcal{O}^1 = \{x \in \mathcal{O} | n(x) = 1\}$.
The normalizer $\mathcal{N}(\mathcal{O}) = \{x \in A^* | x\mathcal{O}x^{-1} = \mathcal{O}\}$.

$\mathcal{O}^1 \subset \mathcal{N}(\mathcal{O})$.

Arithmetic hyperbolic 3-manifolds

Let K be a number field with a pair of complex places, and $n-2$ real places, $A = A_K$ is a quaternion algebra over K which is ramified at every real place of K.

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Theorem: Maclachlan-Reid's Book

Let $\mathcal{O}$ be an order in A, $P\pi\phi(\mathcal{O}^1) < PSL_2(\mathbb{C})$ is a discrete subgroup with finite covolume, i.e. $M = \mathbb{H}^3 / P\pi\phi(\mathcal{O}^1)$ is a finite volume hyperbolic 3-orbifold.

Definition

A hyperbolic orbifold N which is commensurable to any M as above is an arithmetic hyperbolic 3-orbifold.

The field K is called the invariant trace field of N (A. Reid) or the field of definition of the orbifold N (E. B. Vinberg).

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$\mathbb{H}^2/PSL_2(\mathbb{Z})$ is the modular surface, a sphere with three cone points with angles $\{\frac{2\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{\infty}\}$.

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$\mathbb{H}^3/PSL_2(R_{\mathbb{Q}(\sqrt{-1})})$ is commensurable to the Whitehead link.

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M is arithmetic with $K = \mathbb{Q}[x]/(x^3 - x^2 + 1)$, and $A = A_K$ with $Ram(A) = \{\textit{the only one real place}, \mathfrak{p}\}$, $\mathfrak{p}$ is the unique prime ideal in R_K with $5R_K = \mathfrak{p}^3$.

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Let M be the $\{(5,-1),(5,-2)\}$ surgery on the Whitehead link.

M is an orientable closed hyperbolic 3-manifold with the smallest volume among all orientable closed hyperbolic 3-manifolds (Gabai-Meyerhoff-Milley, 2009).

M is arithmetic with $K=\mathbb{Q}[x]/(x^3-x^2+1)$, and $A=A_K$ with $Ram(A)=\{\textit{the only one real place}, \mathfrak{p}\}$, $\mathfrak{p}$ is the unique prime ideal in R_K with $5R_K=\mathfrak{p}^3$.

Theorem: Maclachlan-Reid's Book

There is a one-to-one correspondences between commensurable classes of arithmetic hyperbolic 3-orbifolds and quaternion algebras over number fields with the necessary conditions.

Theorem (H. C. Wang, 1972)

For any simple Lie group, not locally isomorphic to $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$, contains only finitely many conjugacy classes of discrete subgroups of bounded covolume.

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Theorem (Gromov, 1981)

$L_{\mathbb{H}^n}(x)$ be the number of Not Necessary Arithmetic hyperbolic n-manifolds without boundary, $n \geq 4$, then

$$L_{\mathbb{H}^n}(x) < xe^{e^{e^{n+x}}}$$

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Theorem, Burger-Gelander-Lubotzky-Mozes, 2002)

For $n \geq 4$, there are $0 < a < b$ depends on n, such that

$$x^{ax} < L_{\mathbb{H}^n}(x) < x^{bx}.$$

There are infinitely many orbifolds of bounded volume with $\mathbb{H}^2$ and $\mathbb{H}^3$ geometry (Teichmüller space and Thurston's Hyperbolic Dehn Surgery Theory).

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$A = A_K$, $\mathcal{O}^1$ be a maximal order in A, $M = \mathbb{H}^3/P\pi\phi(\mathcal{O}^1)$ has volume

$$\frac{|\Delta(K)|^{\frac{3}{2}}\zeta_K(2)\prod_{\mathfrak{p}|\Delta(A)}(N(\mathfrak{p})-1)}{(4\pi^2)^{|K:\mathbb{Q}|-1}}.$$

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Theorem (Borel, 1981)

Let $AL_{\mathbb{H}^3}(x)$ be the number of arithmetic lattices in $PSL(2,\mathbb{C})$ with covolume at most x, then $AL_{\mathbb{H}^3}(x)$ is finite for any $x \in \mathbb{R}_+$.

Theorem (Belolipetsky-Gelander-Lubotzky-Shalev, 2010)

In two dim:

$$\lim_{x\to\infty}\frac{\log AL_{\mathbb{H}^2}(x)}{x\log x}=\frac{1}{2\pi}.$$

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Conjecture (Belolipetsky-Gelander-Lubotzky-Shalev, 2010)

There is a constant c, such that

$$\lim_{x\to\infty} \frac{\log AL_{\mathbb{H}^3}(x)}{x \log x} = c.$$

Distribution of Max lattices with a fixed trace field K

Theorem (Ma, 2015)

Let $MAL_K(x)$ be the number of maximal arithmetic lattices in $PSL(2, \mathbb{C})$ with covolume at most x and with invariant trace field K, then

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Theorem (Ma, 2015)

Let $MAL_K(x)$ be the number of maximal arithmetic lattices in $PSL(2,\mathbb{C})$ with covolume at most x and with invariant trace field K, then

$$\frac{1}{2}\frac{\rho_K x}{\zeta_K(2)} + \frac{x^{\frac{1}{2|K:\mathbb{Q}|}}}{c_K \log x} < MAL_K(x) < c_K x^3 \log^2 x$$

for x large enough, where ρ_K is the residue of the Zeta function $\zeta_K(x)$ of K at 1.

Definition

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Eichler order $\mathcal{E}$ in $A = A_K$: intersection of two maximal orders $\mathcal{O}_1 \cap \mathcal{O}_2$.
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The proof of the distribution

Fix a maximal order $\mathcal{O}$ in its conjugate class.

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For A with $Ram_f(A) = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$, take $\{\mathfrak{q}_1, \mathfrak{q}_2, \cdots \mathfrak{q}_m\} \subset \mathcal{P}_K$ disjoint from $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$, there is an Eichler order $\mathcal{E}$ with level $\mathfrak{q}_1\mathfrak{q}_2 \cdots \mathfrak{q}_m$.

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Lemma, Maclachlan-Reid's Book

$P\pi\phi(\mathcal{N}(\mathcal{E}))$ is a lattices in $PSL_2(\mathbb{C})$ with covolume

$$\frac{|\Delta(K)|^{\frac{3}{2}}\zeta_K(2)}{(4\pi^2)^{|K:\mathbb{Q}|-1}} \frac{\prod_{j=1}^{m}(N(\mathfrak{q}_j)+1)}{2^{m'}} \frac{\prod_{i=1}^{s}(N(\mathfrak{p}_i)-1)}{[R_{f,\infty}^* : (R_f^*)^2][2^{J_1} : J_2]}$$

for some $0 \le m' \le m$.

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Lemma, Maclachlan-Reid's Book

All maximal lattices are the form of $P\pi\phi(\mathcal{N}(\mathcal{E}))$ for an Eichler order $\mathcal{E}$ with square-free level for some conjugate class of maximal order $\mathcal{O}$.

Remark

Not every $P\pi\phi(\mathcal{N}(\mathcal{E}))$ is a maximal lattice, the condition is on the valuations of K by $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$, $\{\mathfrak{q}_1, \mathfrak{q}_2, \cdots \mathfrak{q}_m\}$.

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Outline the proof of the distribution of max lattices

The upper bound: we count the set of disjoint prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$, $\{\mathfrak{q}_1, \mathfrak{q}_2, \cdots \mathfrak{q}_m\}$ of R_K such that the normalized norm is bounded above.

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The lower bound:we choose a set of disjoint prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$, $\{\mathfrak{q}_1, \mathfrak{q}_2, \cdots \mathfrak{q}_m\}$ of R_K such that the valuations are "independently" in some way, this implies the Eichler order with level $\mathfrak{q}_1\mathfrak{q}_2\cdots\mathfrak{q}_m$ in the algebra A with $Ram_f(A) = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_s\}$ gives a maximal lattice, and then counting.

Thanks for listening!