On Jørgensen, Gehring — Martin — Tan and Tan numbers for groups of figure-eight orbifolds

Alexander Masley (joint work with Andrei Vesnin)

Sobolev Institute of Mathematics and Laboratory of Quantum Topology

Second China-Russia Workshop on Knot Theory and Related Topics
Novosibirsk, August, 21, 2015

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \quad \mathrm{tr}(M) = a + d, \quad \|M\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

A matrix $M \in \mathrm{SL}(2,\mathbb{C})$, such that $M \neq \pm \mathrm{I}$, is called

- *elliptic* if $\mathrm{tr}^2(M) \in [0,4)$,
- *parabolic* if $\mathrm{tr}^2(M) = 4$,
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Denote $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm \mathrm{I}\}$.

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An element $g \in \mathrm{PSL}(2,\mathbb{C})$ is called *elliptic*, *parabolic*, or *loxodromic* if so is its representative in $\mathrm{SL}(2,\mathbb{C})$.

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Consider $\mathrm{PSL}(2,\mathbb{C})$ with the quotient topology of the matrix norm $\|\cdot\|$.

Definition

A subgroup G of $\mathrm{PSL}(2,\mathbb{C})$ is said to be *discrete* if G is a discrete set.

Jorgensen numbers and extreme groups

Let $\mathbb{H}^3$ be the Poincaré half-space model of the hyperbolic 3-space, i. e. the set $\{(z,t) \mid z = x+yi \in \mathbb{C}, t > 0\}$ with the metric $ds^2 = (|dz|^2 + dt^2)/t^2$. Identify $\partial\mathbb{H}^3$ with $\overline{\mathbb{C}}$.

The group $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ as the group of all orientation-preserving isometries and on $\overline{\mathbb{C}}$ as the group of all linear fractional transformations.

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Theorem (T. Jørgensen, 1976)

Suppose that elements $f, g \in \mathrm{PSL}(2,\mathbb{C})$ generate a non-elementary discrete group. Then the following inequality holds: $\left|\mathrm{tr}^2(f) - 4\right| + \left|\mathrm{tr}[f,g] - 2\right| \geq 1$.

For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ denote

$$\mathcal{J}(f,g) = \left|\mathrm{tr}^2(f) - 4\right| + \left|\mathrm{tr}[f,g] - 2\right|.$$

Definition

Let G be a two-generated non-elementary discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. The value $\mathcal{J}(G) = \inf_{\langle f,g\rangle = G} \mathcal{J}(f,g)$ is called the *Jørgensen number* of G.

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Theorem (T. Jørgensen, M. Kiikka, 1975)

The only extreme subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are triangle groups $T(2,3,n) = \left\langle f, g \mid f^2 = g^3 = (fg)^n = \mathrm{I} \right\rangle$, where $n \geq 7$ or $n = \infty$.

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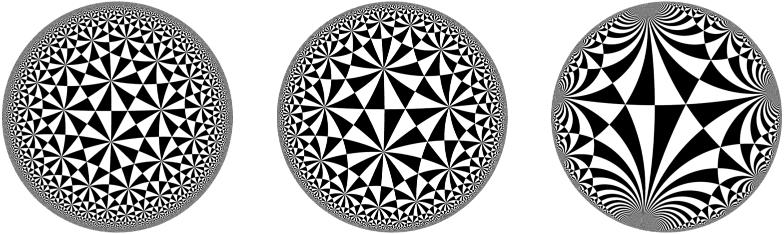

Corollary

The modular group $\mathrm{PSL}(2,\mathbb{Z})$ is extreme.

Proof. It is well known, that $\mathrm{PSL}(2,\mathbb{Z}) = T(2,3,\infty)$. By previous theorem, $T(2,3,\infty)$ is extreme. □

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The Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ is extreme.

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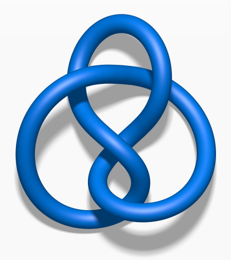
Theorem (M. Oichi, H. Sato, 2006)

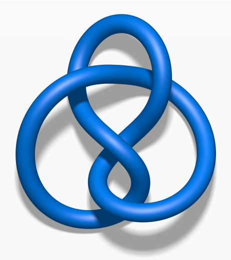
(1) Let $n \in \mathbb{N}$. There exists the group with Jorgensen number equals n.

(2) Let $r \in \mathbb{R}$ and $r > 4$. There exists the group with Jorgensen number equals r.



Let 4_1 be the figure-eight knot in the 3-sphere $\mathbb{S}^3$. The fundamental group $\pi_1(\mathbb{S}^3 \setminus 4_1)$ is called the *figure-eight knot group* and it has the presentation

$$\pi_1(\mathbb{S}^3 \setminus 4_1) = \langle\ f, g \mid [f^{-1}, g]\, f = g\, [f^{-1}, g]\ \rangle.$$



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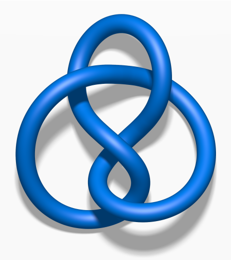
It has faithful representation in $\mathrm{PSL}(2, \mathbb{C})$:

$$f \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ g \to \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix},\ \text{where } \omega = \tfrac{-1+\sqrt{-3}}{2}.$$



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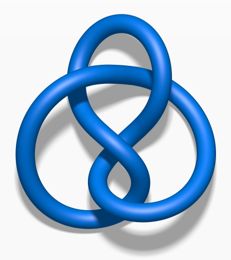

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Proof. By multiplication of matrices, we have

$$[f, g] = fgf^{-1}g^{-1} = \begin{pmatrix} \omega^2 - \omega + 1 & \omega \\ \omega^2 & \omega + 1 \end{pmatrix}.$$

Hence $\mathcal{J}(f, g) = |\,\mathrm{tr}^2(f) - 4| + |\,\mathrm{tr}[f, g] - 2| = |4 - 4| + |\omega^2 + 2 - 2| = 1.$ □

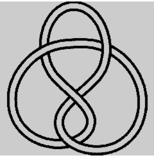
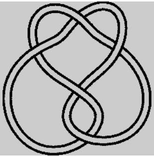
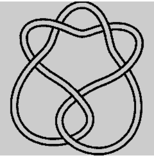
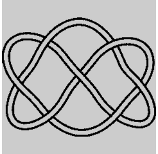
Theorem (J. Callahan, 2009)

Jørgensen numbers of some knot groups are given in the following table:

G	$\mathcal{J}(G)$
$\pi_1(S^3 \setminus 4_1)$	1
$\pi_1(S^3 \setminus 5_2)$	1.32471796...
$\pi_1(S^3 \setminus 6_1)$	1.55603019...
$\pi_1(S^3 \setminus 7_4)$	2.20556943...

Theorem (J. Callahan, 2009)

The figure-eight knot group $\pi_1(S^3 \setminus 4_1)$ is the only torsion-free extreme group.

Let $\mathcal{O}_n$ be the orbifold whose underlying space is the 3-sphere $\mathbb{S}^3$ and whose singular set is the figure-eight knot 4_1 with the isotropy group $\mathbb{Z}_n$. The orbifold group of $\mathcal{O}_n$ has the presentation

$$G_n = \langle\, f, g \mid f^n = g^n = \mathrm{I},\ [f^{-1}, g]\, f = g\, [f^{-1}, g] \,\rangle.$$

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Theorem 1 (A. Masley, A. Vesnin, 2014)

Let G_n be the group of the orbifold $\mathcal{O}_n$ and $n \geq 4$. Then the following two-sided inequality holds:

$$1 \leq \mathcal{J}(G_n) \leq 4\sin^2(\pi/n) + \sqrt{1 + 4\sin^2(\pi/n)}.$$

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Corollary

The following equality holds:

$$\lim_{n\to\infty} \mathcal{J}(G_n) = \mathcal{J}\big(\pi_1(\mathbb{S}^3 \setminus 4_1)\big).$$

GMT-numbers and T-numbers, T-extreme groups and GMT-extreme groups

Suppose that elements $f, g \in \mathrm{PSL}(2,\mathbb{C})$ generate a discrete group. Do there exist constants $a \neq 4$ and $b \neq 2$ such that $\left|\mathrm{tr}^2(f) - a\right| + \left|\mathrm{tr}[f,g] - b\right| \geq 1$?

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Let G be a two-generated discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. The value

$$\mathcal{T}(G) = \inf_{\langle f,g\rangle = G} \mathcal{T}(f,g)$$

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A two-generated discrete group G is said to be *T-extreme* if it can be generated by f and g such that $\mathcal{T}(f,g) = 1$.

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Theorem 2 (A. Masley, A. Vesnin, 2014)

The following two-sided inequality holds:

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Theorem 3 (A. Masley, A. Vesnin, 2014)

For $n \geq 4$, the following two-sided inequality holds:

$$1 \leq \mathcal{T}(G_n) \leq \begin{cases} 3 - 4\sin^2(\pi/n) + \sqrt{3 - 4\sin^2(\pi/n)}, & n \leq 7, \\ 1 + \sqrt{2} + \sqrt{1 + \sqrt{2}}, & n = 8, \\ \sqrt{7 - 8\sin^2(\pi/n)} + \sqrt{3 - 4\sin^2(\pi/n)}, & n \geq 9. \end{cases}$$

Theorem 4 (A. Masley, A. Vesnin, 2014)

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The orbifold group G_4 of the figure-eight knot is a GMT-extreme.

Thank you!