INVARIANTS OF VIRTUAL LINKS

Julia Mikhalchishina

NOVOSIBIRSK – 2015

Braid group

$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ – the braid group.

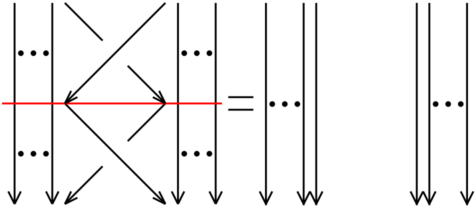
$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Braid group

$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ – the braid group.

$$\begin{array}{rcll} \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } \ i = 1, 2, \dots, n-2, \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for } \ |i-j| \geq 2. \end{array}$$

$$\sigma_i\sigma_i^{-1} = 1$$

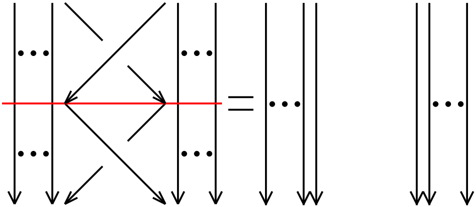


Braid group

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle \text{ – the braid group.}$$

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } \; i = 1, 2, \dots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } \; |i - j| \geq 2. \end{aligned}$$

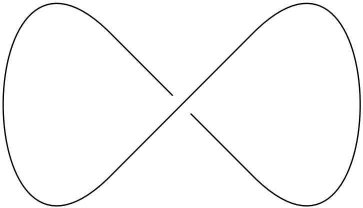
$$\sigma_i \sigma_i^{-1} = 1$$



Example: $(\sigma_2 \sigma_1^{-1})^n$

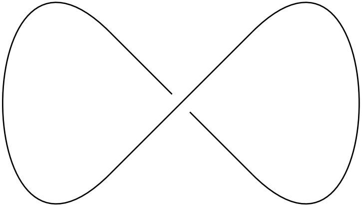
Knots and links

A knot $S^1 \longrightarrow S^3$



Knots and links

A knot $S^1 \longrightarrow S^3$



An n-component link $\underbrace{S^1 \times \ldots \times S^1}_{n} \longrightarrow S^3$

The connection between braids and knots

Alexander's theorem. Given a link L then

$$\exists \beta \in B_n : L = \widehat{\beta}.$$

The connection between braids and knots

Alexander's theorem. Given a link L then

$$\exists \beta \in B_n : L = \widehat{\beta}.$$

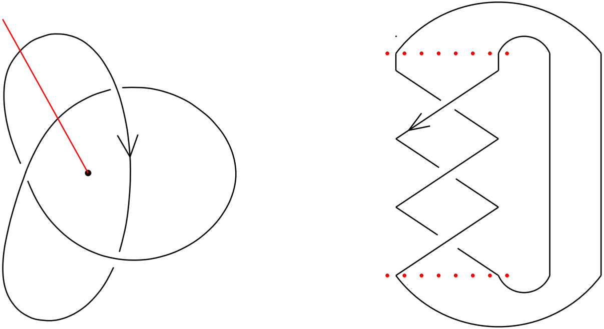


Рис.: The trefoil $T = \widehat{\sigma_1^3}$

Markov's theorem. Given braids $\beta_1, \beta_2 \in B_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{M1, M2} \beta_2$$

$$M1 \qquad \beta \leftrightarrow \sigma_i \beta \sigma_i^{-1} \qquad i = 1, 2, \ldots, n-2, \tag{1}$$

$$M2 \qquad \beta \leftrightarrow \beta \sigma_n^{\pm 1} \qquad \beta \in B_n, \beta \sigma_n^{\pm 1} \in B_{n+1}. \tag{2}$$

Group of the link

$$G(L) = \pi_1(S^3 \backslash N(L)).$$

Group of the link

$$G(L) = \pi_1(S^3 \backslash N(L)).$$

"Braid method". Given $L = \widehat{\beta}$

$$G(L) = \langle x_1, \ldots, x_n \parallel \varphi_A(\beta)(x_i) = x_i,\ i = 1, 2, \ldots, n\rangle,$$

where $\varphi_A : B_n \longrightarrow Aut(F_n)$ – the Artin representation

$$\varphi_A(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1. \end{cases}$$

Virtual braid group

$VB_n = \langle B_n, S_n \rangle = \langle \sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1} \rangle$ – the virtual braid group.

$$I \begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geq 2. \end{cases}$$

Virtual braid group

$VB_n = \langle B_n, S_n \rangle = \langle \sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1} \rangle$ – the virtual braid group.

$$I \begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geq 2. \end{cases}$$

$$II \begin{cases} \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \rho_i \rho_j = \rho_j \rho_i & \text{for } |i-j| \geq 2, \\ \rho_i^2 = 1 & \text{for } i = 1, 2, ..., n-2. \end{cases}$$

Virtual braid group

$VB_n = \langle B_n, S_n \rangle = \langle \sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1} \rangle$ – the virtual braid group.

$$I \left\{ \begin{array}{rcll} \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } \; i = 1, 2, \dots, n-2, \\ \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{for } \; |i-j| \geq 2. \end{array} \right.$$

$$II \left\{ \begin{array}{rcll} \rho_i \rho_{i+1} \rho_i & = & \rho_{i+1} \rho_i \rho_{i+1} & \text{for } \; i = 1, 2, \dots, n-2, \\ \rho_i \rho_j & = & \rho_j \rho_i & \text{for } \; |i-j| \geq 2, \\ \rho_i^2 & = & 1 & \text{for } \; i = 1, 2, ..., n-2. \end{array} \right.$$

$$III \left\{ \begin{array}{rcll} \rho_i \rho_{i+1} \sigma_i & = & \sigma_{i+1} \rho_i \rho_{i+1} & \text{for } \; i = 1, 2, \dots, n-2, \\ \sigma_i \rho_j & = & \rho_j \sigma_i & \text{for } \; |i-j| \geq 2. \end{array} \right.$$

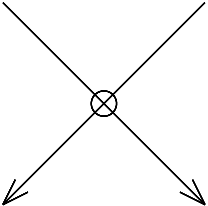
Virtual braid group

$VB_n = \langle B_n, S_n \rangle = \langle \sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1} \rangle$ – the virtual braid group.

$$I \begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geq 2. \end{cases}$$

$$II \begin{cases} \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \rho_i \rho_j = \rho_j \rho_i & \text{for } |i-j| \geq 2, \\ \rho_i^2 = 1 & \text{for } i = 1, 2, ..., n-2. \end{cases}$$

$$III \begin{cases} \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} & \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \rho_j = \rho_j \sigma_i & \text{for } |i-j| \geq 2. \end{cases}$$



Bardakov, Bellingeri; Manturov

$$\psi : VB_n \longrightarrow Aut(F_{n+1}), \quad F_{n+1} = \langle x_1, \ldots, x_n, y \rangle.$$

Bardakov, Bellingeri; Manturov

$$\psi : VB_n \longrightarrow Aut(F_{n+1}), \quad F_{n+1} = \langle x_1, \ldots, x_n, y \rangle.$$

$$\psi|_{B_n} = \varphi_A.$$

$$\psi(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \\ y \mapsto y. \end{cases} \qquad \psi(\rho_i) : \begin{cases} x_i \mapsto y x_{i+1} y^{-1}, \\ x_{i+1} \mapsto y^{-1} x_i y, \\ x_j \mapsto x_j, \ j \neq i, i+1, \\ y \mapsto y. \end{cases}$$

Group of the virtual link

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,$$

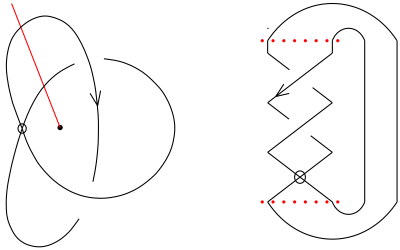
$$G(L_v) = \langle x_1, \ldots, x_n, y \parallel \psi(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

Group of the virtual link

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,$$

$$G(L_v) = \langle x_1, \ldots, x_n, y \parallel \psi(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

Example: The virtual trefoil $T_v = \widehat{\sigma_1^2 \rho_1}$

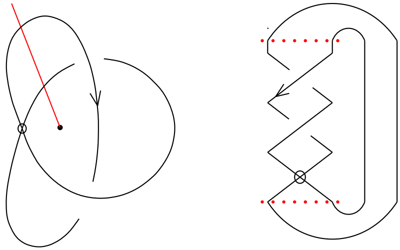


Group of the virtual link

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,$$

$$G(L_v) = \langle x_1, \dots, x_n, y \parallel \psi(\beta_v)(x_i) = x_i,\ i = 1, 2, \dots, n \rangle.$$

Example: The virtual trefoil $T_v = \widehat{\sigma_1^2 \rho_1}$



$$G(T_v) = G(\widehat{\sigma_1^2 \rho_1}) = \langle x_1, x_2, y \parallel \psi(\sigma_1^2 \rho_1)(x_1) = x_1,\ \psi(\sigma_1^2 \rho_1)(x_2) = x_2 \rangle.$$

Wada representations

$$w_1^r, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

$$w_1^r(\sigma_i) : \begin{cases} x_i \to x_i^r x_{i+1} x_i^{-r}, \\ x_{i+1} \to x_i, \\ x_j \to x_j, \text{ for } j \neq i, i+1, \end{cases}$$

$r \in \mathbb{Z}, r \neq 0$.
Note for $r = 1$ this is the Artin representation.

$$w_2(\sigma_i) : \begin{cases} x_i \to x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \to x_i, \\ x_j \to x_j, \text{ for } j \neq i, i+1. \end{cases}$$

$$w_3(\sigma_i) : \begin{cases} x_i \to x_i^2 x_{i+1}, \\ x_{i+1} \to x_{i+1}^{-1} x_i^{-1} x_{i+1}, \\ x_j \to x_j, \text{ for } j \neq i, i+1. \end{cases}$$

Wada representations

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

Wada representations

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

We construct the mappings

$$W_k : VB_n \longrightarrow Aut(F_{n+1}),\ k = 1, 2, 3,$$

$$W_k|_{B_n} = w_k.$$

$$W_k(\rho_i) : \begin{cases} x_i \mapsto y x_{i+1} y^{-1}, \\ x_{i+1} \mapsto y^{-1} x_i y, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1. \end{cases}$$

Wada representations

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

We construct the mappings

$$W_k : VB_n \longrightarrow Aut(F_{n+1}),\ \ k = 1, 2, 3,$$

$$W_k|_{B_n} = w_k.$$

$$W_k(\rho_i) : \begin{cases} x_i \mapsto y x_{i+1} y^{-1}, \\ x_{i+1} \mapsto y^{-1} x_i y, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1. \end{cases}$$

Proposition. *Constructed mappings* W_k, $k = 1, 2, 3$, *are representations of* $VB_n \longrightarrow AutF_{n+1}$.

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,\ k = 1, 2, 3,$$

$$G_k(L_v) = \langle x_1, \ldots, x_n, y \parallel W_k(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,\ k = 1, 2, 3,$$

$$G_k(L_v) = \langle x_1, \ldots, x_n, y \parallel W_k(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

Theorem. Groups $G_k(L_v)$ are invariants of the virtual link L_v, $k = 1, 2, 3$.

Markov Theorem for virtuals

Theorem(Kauffman, Lambropoulou). Given braids $\beta_1, \beta_2 \in VB_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{K1,K2,K3,K4} \beta_2$$

Markov Theorem for virtuals

Theorem(Kauffman, Lambropoulou). Given braids $\beta_1, \beta_2 \in VB_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{K1,K2,K3,K4} \beta_2$$

K1) Virtual and real conjugation: $\rho_k \beta_v \rho_k \sim \beta_v \sim \sigma_k \beta_v \sigma_k^{-1}$,

Markov Theorem for virtuals

Theorem(Kauffman, Lambropoulou). Given braids $\beta_1, \beta_2 \in VB_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{K1,K2,K3,K4} \beta_2$$

K1) Virtual and real conjugation: $\rho_k \beta_v \rho_k \sim \beta_v \sim \sigma_k \beta_v \sigma_k^{-1}$,

K2) Right virtual and real stabilization: $\beta_v \rho_n \sim \beta_v \sim \beta_v \sigma_n^{\pm 1}$,

Markov Theorem for virtuals

Theorem(Kauffman, Lambropoulou). Given braids $\beta_1, \beta_2 \in VB_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{K1,K2,K3,K4} \beta_2$$

K1) Virtual and real conjugation: $\rho_k \beta_v \rho_k \sim \beta_v \sim \sigma_k \beta_v \sigma_k^{-1}$,

K2) Right virtual and real stabilization: $\beta_v \rho_n \sim \beta_v \sim \beta_v \sigma_n^{\pm 1}$,

K3) Algebraic right over/under threading: $\beta_v \sim \beta_v \sigma_n^{\pm 1} \rho_{n-1} \sigma_n^{\mp 1}$,

Markov Theorem for virtuals

Theorem(Kauffman, Lambropoulou). Given braids $\beta_1, \beta_2 \in VB_n$ then

$$\widehat{\beta_1} = \widehat{\beta_2} \Longleftrightarrow \beta_1 \xrightarrow{K1,K2,K3,K4} \beta_2$$

K1) Virtual and real conjugation: $\rho_k \beta_v \rho_k \sim \beta_v \sim \sigma_k \beta_v \sigma_k^{-1}$,

K2) Right virtual and real stabilization: $\beta_v \rho_n \sim \beta_v \sim \beta_v \sigma_n^{\pm 1}$,

K3) Algebraic right over/under threading: $\beta_v \sim \beta_v \sigma_n^{\pm 1} \rho_{n-1} \sigma_n^{\mp 1}$,

K4) Algebraic left over/under threading:

$$\beta_v \sim \beta_v \rho_n \rho_{n-1} \sigma_{n-1}^{\mp 1} \rho_n \sigma_{n-1}^{\pm 1} \rho_{n-1} \rho_n,$$

где $\beta_v, \rho_k, \sigma_k \in VB_n, k = 1, \ldots, n-1$, а $\rho_n, \sigma_n \in VB_{n+1}$.

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

$$\Downarrow$$

$$W_1, W_2, W_3 : VB_n \longrightarrow Aut(F_{n+1}) \text{ [Proposition.]}$$

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

$$\downdownarrows$$

$$W_1, W_2, W_3 : VB_n \longrightarrow Aut(F_{n+1}) \text{ [Proposition.]}$$

$$\downdownarrows$$

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,\ k = 1, 2, 3,$$

$$G_k(L_v) = \langle x_1, \ldots, x_n, y \parallel W_k(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

$$w_1, w_2, w_3 : B_n \longrightarrow Aut(F_n).$$

$$\Downarrow$$

$$W_1, W_2, W_3 : VB_n \longrightarrow Aut(F_{n+1}) \text{ [Proposition.]}$$

$$\Downarrow$$

$$\text{Let } L_v = \widehat{\beta_v},\ \beta_v \in VB_n,\ k = 1, 2, 3,$$

$$G_k(L_v) = \langle x_1, \ldots, x_n, y \parallel W_k(\beta_v)(x_i) = x_i,\ i = 1, 2, \ldots, n \rangle.$$

$$\Downarrow$$

Theorem. Groups $G_k(L_v), k = 1, 2, 3$, are invariants of the virtual link L_v.

特别感谢你们