

Tetrahedral manifolds and links

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Outline:

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2. Tetrahedral manifolds.
3. Arithmeticity of tetrahedral manifolds.
4. Tetrahedral links.
5. Nice descriptions for infinite families.

1. Cusped hyperbolic 3-manifolds.

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Knot and link complements arise as examples of cusped manifolds.

We say that **complexity** $c(M)$ of a cusped hyperbolic 3-manifold M is equal to k if M admits an ideal triangulation with k tetrahedra and there is no an ideal triangulation with less number of tetrahedra.

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Possible approach:

Classify cusped hyperbolic 3-manifolds according to their complexity.

Cusped hyperbolic manifolds of complexity ≤ 8 .

[P. Callahan – M. Hildebrand – J. Weeks, 1999]

All 4,815 hyperbolic 3-manifolds which can be glued with ≤ 7 ideal tetrahedra.

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[P. Callahan – J. Dean – J. Weeks, 1999],[A. Champanerkar – I. Kofman – T. Mullen, 2014]

All hyperbolic knot complements which can be glued with at most 8 ideal tetrahedra.

Tetrahedra	1	2	3	4	5	6	7	8	≤ 8
1-Cusped Manifolds	0	2	9	52	223	913	3388	12241	16828
Knots	0	1	2	4	22	43	129	301	502

Hyperbolic knot complements of small complexity.

$k=2$:



$k=3$:



$k=4$:



2. Tetrahedral manifolds.

We call a cusped hyperbolic 3-manifold **tetrahedral** if it can be decomposed into **regular ideal tetrahedra**. Let M be a tetrahedral manifold which can be decomposed into k regular ideal tetrahedra. Since regular ideal tetrahedron has maximal volume,

$$c(M) = k.$$

For $k = 1$ there is a unique tetrahedral manifold – **H. Gieseking [1912]**, non-orientable.

For $k = 2$ one of two orientable tetrahedral manifolds is the figure-eight knot complement.

[P. Callahan, M. Hildebrand, J. Weeks]: listed all 4,815 hyperbolic 3-manifolds which can be glued from ≤ 7 ideal (not necessary regular) tetrahedra.

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[Fominykh – Tarkaev – V., 2014]: independent generation of orientable tetrahedral manifolds of complexity at most 8. Recognition: by homology and Turaev – Viro quantum invariants of 3-manifolds.

Theorem.

There are only 29 orientable tetrahedral manifolds of complexity at most 8. Among them 17 have 1 cusp and 12 have 2 cusps.

[E. Fominykh – S. Garoufalidis – M. Goerner – V. Tarkaev – V., arXiv:1502.00383]

The list of all tetrahedral hyperbolic manifolds up to 25 tetrahedra for **orientable** case and up to 21 tetrahedra for **non-orientable** case is obtained.

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You will find also 62 source and data files for **SnapPy** and **Regina** attached.

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The proof use **canonical cell decomposition of cusped manifolds** introduced by Epstein and Penner and associate to each manifold its **isometry signature**.

- [SnapPy](#) is a modern user interface to the Jeff Weeks's SnapPea kernel. SnapPy combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. Can be used under the [Sage](#). Project by Marc Culler and Nathan Dunfield.

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- **Regina** is a program for studying 3-manifolds including support for normal surfaces and angle structures. Project by Ben Burton.
- **Recognizer** is a program for studying 3-manifolds including support for spines. Project by Sergei Matveev's group.

Number of tetrahedral manifolds of given complexity (1 – 13 tet.).

Tet.	Or. mfls	Non-or. mfls
1	0	1
2	2	1
3	0	1
4	4	2
5	2	8
6	7	10
7	1	1
8	13	6
9	1	6
10	47	197
11	0	17
12	47	80
13	3	8

Number of tetrahedral manifolds of given complexity (14 – 25 tet.).

Tet.	Or. mflds	Non-or. mflds
14	58	113
15	81	822
16	96	142
17	8	52
18	199	810
19	25	326
20	1684	22340
21	31	251
22	381	-
23	58	-
24	1465	-
25	7367	-
Total	11580	25194

All orientable tetrahedral triangulations with $n \leq 7$ tetrahedra.

n	Signatures	Name	n	Signatures	Name
2	cPcbbbdxm	02 ₀₀₀₀	6	gLLPQccdfefqjsqjj	06 ₀₀₀₀
2	cPcbbbiht	02 ₀₀₀₁	6	gLLPQccdfeffqjsqsj	06 ₀₀₀₁
4	eLMkbbdddemdx	04 ₀₀₀₀	6	gLLPQceefffpupu	06 ₀₀₀₂
4	eLMkbcdddeded	04 ₀₀₀₁	6	gLMzQbcdefffhxqxha	06 ₀₀₀₃
4	eLMkbcdddhxqdu	04 ₀₀₀₂	6	gLMzQbcdefffhxqxxq	06 ₀₀₀₄
4	eLMkbcdddhxqlm	04 ₀₀₀₃	6	gLvQQadfedefjqjasjj	06 ₀₀₀₅
5	fLLQcbcedeeloxset	05 ₀₀₀₀	6	gLvQQbefeefedimipt	06 ₀₀₀₆
5	fLLQcbdeedemnamjp	05 ₀₀₀₁	7	hLvAQkadfdgggfjxqnjnbw	07 ₀₀₀₀

For any triangulation \mathcal{T} with n tetrahedra we find **isomorphism signature** that is the lexicographically smallest $24n$ bit sequence presenting the triangulation. We use 64-version of the dehydration presentation of the sequence taking in account the correspondence between integers $0, 1, \dots, 63$ and characters:

integer	0	...	25	26	...	51	52	...	61	62	63
character	a	...	z	A	...	Z	0	...	9	+	-

- install SnapPy
- `cd /Users/user/Desktop/SnapPy_data/tetrahedralCensus/snappy/`
- `import sys; sys.path.append("")`
- `from tetrahedralCuspedCensus import *`
- `M=TetrahedralOrientableCuspedCensus['otet25_7366']`
- `M.browse()`

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Manifold $otet25_{7366}$

Volume: 25.373540160

Chern–Simons Invariant: -0.1250000

First Homology: $\mathbb{Z}/2 + \mathbb{Z}/2 + \mathbb{Z} + \mathbb{Z}$

Fundamental Group has 4 generators and 3 relations

Symmetry group is trivial

Dirichlet domain

etc.

3. Arithmeticity of tetrahedral manifolds.

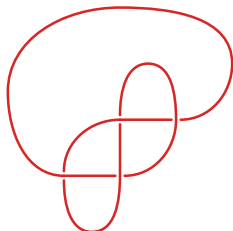
The group

$$\Gamma = \pi_1(S^3 \setminus \mathcal{F}) = \langle a, b \mid ab^{-1}aba^{-1} = b^{-1}aba^{-1}b \rangle.$$

has a faithful presentation in $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}^+ \mathbb{H}^3$:

$$\theta(a) \mapsto \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \quad \theta(b) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $\omega = -1/2 + \sqrt{-3}/2$. Hence $\Gamma \subset \mathrm{PSL}(2, \mathbb{Q}(\sqrt{-3}))$.



Let $F = \mathbb{Q}(\sqrt{-d})$, $d \geq 1$, and \mathcal{O}_d be the ring of integers of F .

$\mathrm{PSL}(2, \mathcal{O}_d)$ – **Bianchi group** – is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

Thus, $\mathbb{H}^3/\mathrm{PSL}(2, \mathcal{O}_d)$ is a hyperbolic 3-orbifold.

If G is a torsion-free subgroup of $\mathrm{PSL}(2, \mathcal{O}_d)$ then \mathbb{H}^3/G is an orientable hyperbolic 3-manifold.

[R. Riley, 1975]

The group $\Gamma = \pi_1(S^3 \setminus 4_1)$, where 4_1 is the figure-eight knot, is a subgroup of $\mathrm{PSL}(2, \mathcal{O}_3)$ of index 12.

[A. Reid, 1991]

The figure-eight knot is the only one knot whose group is arithmetic.

Two manifolds (or orbifolds) are **commensurable** if they have a common finite cover.

Lemma [due to C. Maclachlan and A. Reid]

For a cusped hyperbolic manifold M the following are equivalent

- M is commensurable with the figure-eight knot complement.
- M is arithmetic with invariant trace field $\mathbb{Q}(\sqrt{-3})$.
- The invariant trace field of M is $\mathbb{Q}(\sqrt{-3})$ and M has integer traces.

Lemma.

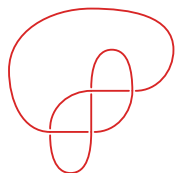
All tetrahedral manifolds are **arithmetic** with invariant trace field $\mathbb{Q}(\sqrt{-3})$ and **commensurable** to each other.

Remark. The converse doesn't hold. There are (at least 8) **arithmetic** manifolds **commensurable** to the figure-eight knot complement which are **non tetrahedral**.

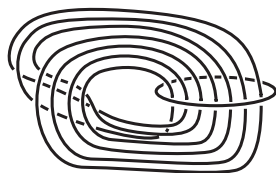
4. Tetrahedral links.

Tetrahedral links with 2, 4 and 8 tetrahedra.

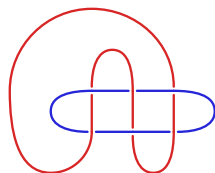
A link L is called **tetrahedral** if $S^3 \setminus L$ is a tetrahedral hyperbolic manifold.



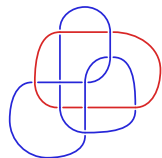
02₀₀₀₁($K4a1$)



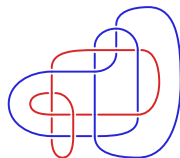
04₀₀₀₀



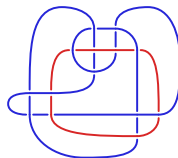
04₀₀₀₁($L6a2$)



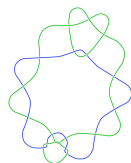
08₀₀₀₂($L10n46$)



08₀₀₀₉($L14n38547$)

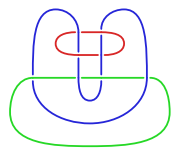


08₀₀₀₁($L14n24613$)

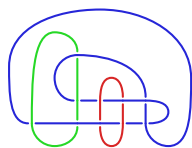


08₀₀₀₅

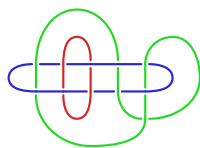
Tetrahedral links with 10 tetrahedra.



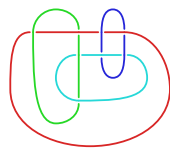
10₀₀₀₆(*L8a20*)



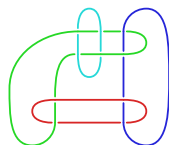
10₀₀₄₂(*L10n88*)



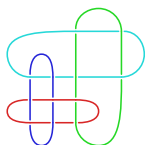
10₀₀₀₈(*L11n354*)



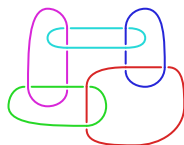
10₀₀₁₁(*L8a21*)



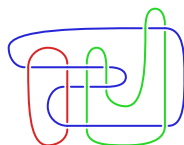
10₀₀₁₄(*L10n101*)



10₀₀₂₈(*L12n2201*)

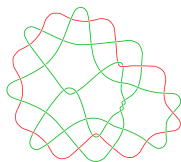


10₀₀₂₇(*L10n113*)

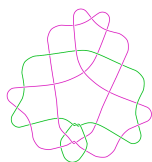


10₀₀₄₃(*L12n1739*)

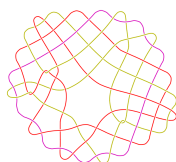
Tetrahedral links with 10 and 12 tetrahedra.



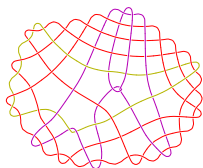
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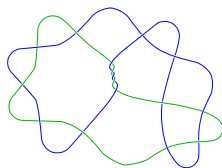
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100025

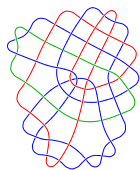


120001

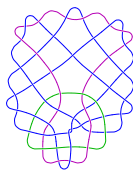


120005

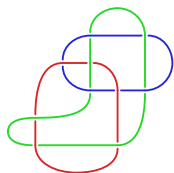
Tetrahedral links with 12 tetrahedra.



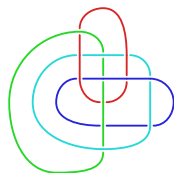
12₀₀₀₆



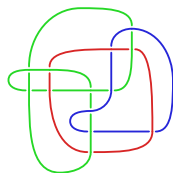
12₀₀₁₀



12₀₀₀₇(*L10a157*)



12₀₀₀₉(*L12n2208*)



12₀₀₁₈(*L13n9382*)

We say that M is a **homology link complement** if $H_1(M, \mathbb{Z}) = \mathbb{Z}^c$, where c is the number of cusps. This condition is equivalent for M to be a link complement in an integer homology sphere.

Tetrahedra	homology links
2	1 (K4a1)
4	2 (Berge link, L6a2)
8	5 (4 of them are L10n46, L14n38547, L14n24613, otet8 ₀₀₀₅)
10	12 (11 of them are L8a20, L10n88, L11n354, L8a21, L10n101, L12n2201, L10n113, L12n1739, otet10 ₀₀₀₇ , otet10 ₀₀₀₃ , otet10 ₀₀₂₅)
12	7
14	25
16	32
18	66
20	209
22	148
24	378

5. Nice descriptions for infinite families.

Remark.

If N is an n -fold covering of a tetrahedral manifold M then N is also tetrahedral and

$$c(N) = n \cdot k.$$

It gives infinite families of cusped manifolds with known complexity.

The firstly this idea was realized by S. Anisov for coverings of the figure-eight knot complement.

[S. Anisov, 2005]: Let N_n be the total space of the punctures torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n$.

N_n is the n -fold cover of the figure-eight knot complement N_1 .

[S. Anisov, 2005]

N_n can be decomposed in $2n$ ideal regular tetrahedra.

[A. Champanerkar, I. Kofman, J. Purcell, [arxiv:1506.04139](https://arxiv.org/abs/1506.04139)]:

The weaving knot $W(p, q)$ is the alternating knot or link with the same projection as the standard p -braid $(\sigma_1 \sigma_2 \dots \sigma_{p-1})^q$ projection of the torus knot or link $T(p, q)$.

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$W(3, 2)$ is the figure-eight knot, the closer of the braid $(\sigma_1 \sigma_2^{-1})^2$.

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Let B be a axis of the q -rotation symmetry of $W(p, q)$.

$W(3, 1) \cup B$ is the link 6_2^2 .

[A. Champanerkar, I. Kofman, J. Purcell, arxiv:1506.04139]:

The weaving knot $W(p, q)$ is the alternating knot or link with the same projection as the standard p -braid $(\sigma_1 \sigma_2 \dots \sigma_{p-1})^q$ projection of the torus knot or link $T(p, q)$.

$W(3, 2)$ is the figure-eight knot, the closer of the braid $(\sigma_1 \sigma_2^{-1})^2$.

Let B be a axis of the q -rotation symmetry of $W(p, q)$.

$W(3, 1) \cup B$ is the link 6_2^2 .

[A. Champanerkar, I. Kofman, J. Purcell, July 2015]

A manifold $S^3 - (W(3, 1) \cup B)$ admit a decomposition in 4 ideal regular tetrahedra.

A manifold $S^3 - (W(3, q) \cup B)$ admit a decomposition in $4q$ ideal regular tetrahedra.