Polynomial automorphisms of $\mathbf{C}^n$ preserving Markoff-Hurwitz polynomial

Ying Zhang
Soochow University

Joint work with H.Hu and S.P.Tan in [arXiv1501.06955v2]

2nd China-Russia Workshop on Knot Theory & Related Topics
August 21-25, 2015, Novosibirsk, Russia

Abstract

We study the **action** of the group of polynomial automorphisms on $\mathbf{C}^n$ ($n \geq 3$) preserving the Markoff-Hurwitz polynomial

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_1 x_2 \cdots x_n.$$

Our main results include

1. **determination of the group**,
2. description of **domain of discontinuity** (a non-empty open subset of $\mathbf{C}^n$ on which the group acts properly discontinuously),
3. **identities** for the orbit of a point in the domain of discontinuity.

Background: $n = 3$ Markoff equation

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 0,$$

or **in number theoretic form**:

$$a^2 + b^2 + c^2 - 3abc = 0,$$

with $x_1 = 3a$, $x_2 = 3b$ and $x_3 = 3c$. Integral solutions:

$$(a, b, c) = (1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), \cdots$$

Related to **Diophantine approximation** - A.A.Markoff, 1880

Related to **hyperbolic geometry** - H.Cohn, 1955, Ann. Math.

Some polyn. autom. of $\mathbf{C}^n$ preserving the M-H polynomial

Three classes of polynomial automorphisms of $\mathbf{C}^n$

1. **permutations** of coordinates,
2. **even-sign-changes** of coordinates, e.g.

$$(x_1, x_2, x_3, x_4) \longmapsto (-x_1, -x_2, x_3, x_4).$$

3. **Vieta involutions**, e.g.

$$(x_1, x_2, x_3, x_4) \longmapsto (x_2x_3x_4 - x_1, x_2, x_3, x_4).$$

Note The automorphisms in classes 1 and 2 are **linear**.

Determination of the groups

Let $\mathbf{\Gamma_n^*}$ be the group of polynomial automorphisms of $\mathbf{C}^n$ preserving the Markoff-Hurwitz polynomial $x_1^2 + x_2^2 + \cdots + x_n^2 - x_1 x_2 \cdots x_n$.

Let $\mathbf{\Gamma_n} < \Gamma_n^*$ be gen. by the Vieta involutions. Then $\mathbf{\Gamma_n} \lhd \Gamma_n^*$.

Theorem (Hu-Tan-Z)
$\mathbf{\Gamma_n^*}$ is generated by the permutations, the even-sign-changes, and the Vieta involutions. Furthermore, the index $[\mathbf{\Gamma_n^*} : \mathbf{\Gamma_n}]$ is finite.

Note
Case n=3 R.D. Horowitz, 1975, Trans.AMS

Cayley graph of Γ_n

Let $\boldsymbol{\Gamma_n} = (\mathbf{Z}_2)^{*n}$ be the **group given by presentation**

$$\Gamma_n = \langle\, b_1, \cdots, b_n \mid b_1^2 = 1, \cdots, b_n^2 = 1 \,\rangle.$$

Let $\boldsymbol{\Delta}$ be the **Cayley graph** of Γ_n. Then

- Δ is an infinite ***n*-valent tree**, and
- Edges are **colored** by $1, \cdots, n$ (corresponding to $b_1, \cdots, b_n$).

Explicitly, $V(\Delta) = \Gamma_n$ and $E(\Delta) = \bigsqcup_{i=1}^{n} E_i(\Delta)$,

$$E_i(\Delta) = \{(g, g') \in \Gamma_n \times \Gamma_n \mid g' = gb_i\}.$$

Γ_n acts on $\mathbf{C}^n$

Γ_n acts on $\mathbf{C}^n$ with $b_1, \cdots, b_n$ acting as Vieta involutions:

$$b_i(x_1, \cdots, x_n) = (x'_1, \cdots, x'_n)$$

where

$$x'_j = x_j$$

for all indices $j \in [n] \backslash \{i\}$ and

$$x'_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n - x_i.$$

Vector-valued Markoff-Hurwitz map

- $z \in \mathbf{C}^n$ induces a **vector-valued Markoff-Hurwitz map**

$$\varphi = \varphi_z : V(\Delta) \equiv \Gamma_n \to \mathbf{C}^n$$

$$\varphi(g) = g(z).$$

- $\varphi = \varphi_z$ satisfies the **edge relations**:

for each edge $e = (g, g') \in E(\Delta)$ with $g' = g\, b_i$

$$x_i + x_i' \;=\; x_1 \cdots x_{i-1}\, x_{i+1} \cdots x_n,$$

where $x = \varphi(g) \in \mathbf{C}^n$ and $x' = \varphi(g') \in \mathbf{C}^n$.

Regular (= Regularly labeled) of subtrees Δ

$T^I(v)$:= the **regular subtree** of Δ, with color set I and containing the vertex v

$\mathbf{T}^I(\Delta) := \{$ regular subtrees of Δ with color set I $\}$

$\mathbf{T}^{(k)}(\Delta) := \{$ regular k-subtrees of Δ $\}$

Decomposition $\mathbf{T}^{(k)}(\Delta) = \bigsqcup_{I \in \mathcal{P}(n;k)} \mathbf{T}^I(\Delta)$.

$\mathcal{P}(n;k)$:= the set of all k-element-subsets of $[n]$

$[n] := \{1, \cdots, n\}$, the set of colors

Regular k-subtrees: $k = 0, 1, 2, n-1$

$\mathbf{T}^{(0)}(\Delta) = V(\Delta)$, the set of vertices

$\mathbf{T}^{(1)}(\Delta) = E(\Delta)$, the set of edges

$\mathbf{T}^{(2)}(\Delta) =: \mathcal{A}$, the set of **alternating geodesics**

$\mathbf{T}^{(n-1)}(\Delta) =: \mathcal{X}$, the set of **regions**

$\mathbf{T}^{[n]\setminus\{i\}}(\Delta) =: \mathcal{X}_i$, the set of **decolor i regions**

Regular k-subtrees; Vertices

Every regular k-subtree is the intersection of $n - k$ regular $(n-1)$-subtrees:

$$T^I(v) = \bigcap_{i \neq i_1, \cdots, i_k} T^{[n]\setminus\{i\}}(v) = \bigcap_{i \neq i_1, \cdots, i_k} X_i.$$

with $I = \{i_1, \cdots, i_k\} \subset [n]$ and $X_i \in \mathcal{X}_i$.

A vertex is the intersection of n regular $(n-1)$-subtrees:

$$v = \bigcap_{i \in [n]} T^{[n]\setminus\{i\}}(v) = \bigcap_{i \in [n]} X_i, \quad X_i \in \mathcal{X}_i.$$

Edges of Δ

A color i edge $e \in E_i(\Delta)$ can be written as

$$e = \bigcap_{j \neq i} X_j, \quad X_j \in \mathcal{X}_j.$$

If $e = (v, v')$ where $v = X_i \cap \bigcap_{j \neq i} X_j$ and $v' = X_i' \cap \bigcap_{j \neq i} X_j$, then we write

$$\begin{aligned} e &\longleftrightarrow (\{X_1, \cdots, \hat{X}_i, \cdots, X_n\}; \{X_i, X_i'\}), \\ \vec{e} &\longleftrightarrow (\{X_1, \cdots, \hat{X}_i, \cdots, X_n\}; X_i \to X_i') \end{aligned}$$

for the directed edge $\vec{e} = (v \to v')$.

Hurwitz map & Edge relations

A **Hurwitz map** is a function $\phi : \mathcal{X} \equiv \mathbf{T}^{(n-1)}(\Delta) \to \mathbf{C}$ which satisfies the **edge relations**:

If $e \leftrightarrow (\{X_1, \cdots, \hat{X}_i, \cdots, X_n\}; \{X_i, X_i'\})$ then

$$\phi(X_i) + \phi(X_i') = \prod_{j \in [n] \setminus \{i\}} \phi(X_j)$$

or, writing $x_i = \phi(X_i)$ and $x_i' = \phi(X_i')$ etc., in the form

$$x_i + x_i' = x_1 \cdots x_{i-1} \, x_{i+1} \cdots x_n .$$

Notation $\boldsymbol{\Phi} :=$ the set of all Hurwitz maps

Hurwitz map & Vertex relations

A Hurwitz map $\phi : \mathcal{X} \to \mathbf{C}$ also satisfies the **vertex relations**:

there exists $\mu = \mu(\phi) \in \mathbf{C}$ such that at each vertex v,

$$x_1^2 + \cdots + x_n^2 - x_1 \cdots x_n = \mu,$$

where $\phi(v) = (x_1, \cdots, x_n) \in \mathbf{C}^n$.

Notation Given $\mu \in \mathbf{C}$, we write

$$\mathbf{\Phi}_\mu := \{\phi \in \mathbf{\Phi} \mid \mu(\phi) = \mu\}.$$

Hurwitz maps extended

A Hurwitz map $\phi : \mathcal{X} \equiv \mathbf{T}^{(n-1)}(\Delta) \to \mathbf{C}$ **extends** as $\phi : \mathbf{T}^{(k)}(\Delta) \to \mathbf{C}$ for $0 \le k < n-1$, defined by

$$\phi\Big(\bigcap_{j \neq i_1, \cdots, i_k} X_j \Big) = \prod_{j \neq i_1, \cdots, i_k} \phi(X_j).$$

In particular, with notation used above,

$$\begin{aligned} \phi(v) &= x_1 \cdots x_n, & \\ \phi(e) &= \prod_{m \neq i} x_m, & e \in E_i(\Delta) \\ \phi(\gamma) &= \prod_{m \neq i,j} x_m, & \gamma \in \mathcal{A}_{\{i,j\}}. \end{aligned}$$

Bowditch set $\mathcal{B} \subset \mathbf{\Phi}$

The **Bowditch set** $\mathcal{B} \subset \mathbf{\Phi}$ consists of those Hurwitz maps

$$\phi : \mathcal{X} \equiv \mathbf{T}^{(n-1)}(\Delta) \to \mathbf{C}$$

which satisfy conditions (B1) and (B2) below:

(**B1**) $\forall\, \gamma \in \mathcal{A} \equiv \mathbf{T}^{(2)}(\Delta)$, $\phi(\gamma) \notin [-2, 2] \subset \mathbf{R}$.

(**B2**) $\exists\, K > 2$ such that $\{\gamma \in \mathcal{A};\ |\phi(\gamma)| \leq K\}$ **is finite**.

Remark If (B2) holds for **some** $K > 2$ then it holds for **all** $K > 0$.

Notation $\mathcal{B}_\mu = \mathcal{B} \cap \mathbf{\Phi}_\mu$

Bowditch set is open

We **identify** $\mathbf{\Phi} \longleftrightarrow \mathbf{C}^n$ **by**

$$\phi \longleftrightarrow (\phi(Z_1), \cdots, \phi(Z_n))$$

where vertex $v_0 = \bigcap_{i\in[n]} Z_i$ with $Z_i \in \mathcal{X}_i$ corresponds to $g_0 \in \Gamma_n$.

Theorem (Hu-Tan-Z) **Bowditch set** $\mathcal{B} \subset \mathbf{\Phi}$ **is open**.

Case n=3
For $\mu = 0$: Bowditch, 1998, Proc.LMS
For $\mu \in \mathbf{C}$: Tan-Wong-Z, 2008, Adv.Math.

ϕ-Induced directed tree $\vec{\Delta}_\phi$

Given a Hurwitz map ϕ, we **direct** each edge $e \in E_i(\Delta)$ where $e \leftrightarrow (\{X_1, \cdots, \hat{X}_i, \cdots, X_n\}; \{X_i, X_i'\})$ as follows:

If $|\phi(X_i)| \neq |\phi(X_i')|$, we direct e **decisively**

$$\begin{array}{ll} \text{from } X_i \text{ to } X_i' & \text{if } |\phi(X_i)| > |\phi(X_i')|; \\ \text{from } X_i' \text{ to } X_i & \text{if } |\phi(X_i')| > |\phi(X_i)|. \end{array}$$

If $|\phi(X_i)| = |\phi(X_i')|$, we direct e **arbitrarily**.

$\vec{\Delta}_\phi$:= the resulting directed tree.

Aim to show: For $\phi \in \mathcal{B}$, $\exists$ a **finite subtree** T_ϕ of Δ such that each edge not in T_ϕ is ϕ-directed towards T_ϕ decisively.

Fibonacci function $F : \mathcal{A} \to \mathbb{N}$

We define a **function** $F : \mathcal{A} \to \mathbb{N}$ relative to a vertex $v_0 \in V(\Delta)$ recursively by induction on $d(\gamma)$, the distance of γ from v_0.

If $d(\gamma) = 0$, that is, $v_0 \in \gamma$, we define $F(\gamma) = 1$.

If $d(\gamma) > 0$, let v^* be the unique vertex in γ which is closest to v_0;
then $\gamma = T^{\{i,j\}}(v^*)$.
Let $e^* \in E_k(\Delta)$ be the unique edge incident to v^* which lies on the geodesic from v^* to v_0. Then $k \neq i, j$.

Let $\alpha = T^{\{i,k\}}(v^*)$, $\beta = T^{\{j,k\}}(v^*)$. Then $d(\alpha), d(\beta) < d(\gamma)$.

We define

$$F(\gamma) = F(\alpha) + F(\beta).$$

Convergence via Fibonacci growth

- **Convergence of a series**

Proposition If $s > n - 1$ then

$$\sum_{\gamma \in \mathcal{A}} (F(\gamma))^{-s} < +\infty.$$

- **Fibonacci growth of** $\log^+ |\phi(\cdot)| := \max\{\log |\phi(\cdot)|, 0\}$

Proposition If $\phi \in \mathcal{B}$ then $\log^+ |\phi(\cdot)| : \mathcal{A} \to \mathbf{R}_{\geq 0}$ has Fibonacci growth, that is, there exist $0 < c_1 < c_2$ such that

$$c_1 F(\gamma) \leq \log^+ |\phi(\gamma)| \leq c_2 F(\gamma)$$

except for possibly a finite number of elements $\gamma \in \mathcal{A}$.

ϕ-Attracting subtree

Given $\phi \in \mathbf{\Phi}$, we say that a subtree T of Δ is **ϕ-attracting** if for each edge e not in T, the ϕ-directed edge on e is directed decisively towards T.

Proposition If $\phi \in \mathcal{B}$ is such that $|\phi(\gamma)| \geq K$ for some $K > 2$ and for all $\gamma \in \mathcal{A}$, then there is a unique ϕ-attracting vertex.

Recall that, for a general $\phi \in \mathcal{B}$, there exists $K > 0$ such that $|\phi(\gamma)| \leq K$ for only finitely many $\gamma \in \mathcal{A}$.

We **construct** a **ϕ-attracting finite subtree** T_ϕ by taking the union of non-empty, ϕ-attracting finite sub-intervals $J(\gamma)$ of $\gamma \in \mathcal{A}$ for all γ with $|\phi(\gamma)| \leq K$.

Domain of discontinuity

Let us write

$$\mathbf{B} \longleftrightarrow \mathcal{B}$$

under the identification $\mathbf{C}^n \longleftrightarrow \mathbf{\Phi}$.

Theorem (Hu-Tan-Z) $\mathbf{\Gamma_n}$ acts on **B properly discontinuously**.

Remark However, the boundary of **B** should be fractal.

Convergence of infinite sums

Proposition If $\phi \in \mathcal{B}$, then for all $t > 0$, the infinite sum

$$\sum_{v \in V(\Delta)} |\phi(v)|^{-t} < +\infty.$$

Proposition If $\phi \in \mathcal{B}$, then for all $t > 0$, the infinite sum

$$\sum_{X \in \mathcal{X}} |\phi(X)|^{-t} < +\infty.$$

Identity for $\phi \in \mathcal{B}$

Theorem (Hu-Tan-Z) If $\phi \in \mathcal{B}$, the following **identity** holds:

$$\sum_{\gamma \in \mathcal{A}} \left(1 - \sqrt{1 - \frac{4}{\phi(\gamma)^2}} \right) - \sum_{v \in V(\Delta)} \frac{\mu}{\phi(v)} = 1,$$

where each of the two infinite sums converges absolutely.

Remark In the identity, the square root $\sqrt{z}$ for $z \in \mathbf{C} - \{0\}$ is required to have positive real part.

Case $n = 3$**,** $\mu = 0$: McShane identity (Bowditch's version)

Identity Reformed

Theorem (Hu-Tan-Z) If $\phi \in \mathcal{B}$, the following **identity** holds:

$$\sum_{\gamma \in \mathcal{A}} \left(1 - \left(1 + \frac{2\mu}{n(n-1)(\sigma(\gamma) - \mu)} \right) \sqrt{1 - \frac{4}{\phi(\gamma)^2}} \right) = 1,$$

where the infinite sum converges absolutely.

Notation For $\gamma \in \mathcal{A}^{\{i,j\}}$, we write $\sigma(\gamma) := \sum_{m \neq i,j} x_m^2$.

Ideas of proof

- ϕ-Induced weight
- A finite identity for a circular set
- Passing to the limit
- Reforming the identity

The End

Thanks!