VOLUMES OF NON-EUCLIDEAN POLYHEDRA

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ABSTRACT. In this talk we overview the volume calculations for polyhedra in constant curvature spaces. We also study the problem in which cases there exist a circumscribed hyperbolic sphere or orisphere or equidistant surface for the hyperbolic simplex. We give necessary and sufficient condition for each of three possible situations.

1. INTRODUCTION

The calculation of volumes for polyhedra in spaces of constant curvature is very old and difficult problem. It is of special interest because any 3-manifold can be considered as a polyhedron with pairwise identified faces. By Thurston, a wide class of 3-manifolds can be endowed with a complete metric of negative curvature. In particular, it turned out that the complement of a simple knot (excepting torus and satellite ones) admits a hyperbolic structure. These facts and Mostow Rigidity theorem allow to consider the hyperbolic volume as one of the most important invariants of manifolds and polyhedra.

Since Lobachevsky and Schläfli (see [1] and [2] respectively) the volume formula for biorthogonal tetrahedron (so called orthoscheme) is known. The volume of the Lambert cube and some other polyhedra were calculated by R. Kellerhals [3], D. A. Derevnin, A. D. Mednykh [4], A. D. Mednykh, J. Parker, A. Yu. Vesnin [5] and others. The volume of ideal hyperbolic polyhedra in many important particular cases was obtained by E. B. Vinberg [6]. The general formula for volume of tetrahedron remained to be unknown for a long time. A few years ago Y. Choi, H. Kim [7], J. Murakami, U. Yano [8] and A. Ushijima [9] were succeeded in finding of a such formula. D. A. Derevnin, A. D. Mednykh [10] suggested an elementary integral formula for the volume of hyperbolic tetrahedron. It was discovered by J. Milnor [11] and by D. A. Derevnin, A. D. Mednykh and M. G. Pashkevich [12] that in the case when all faces of tetrahedron are mutually congruent the volume formula can be obtained in a very explicit way. Surprisedly, but in 1906 an essential advance in volume calculation was achieved by Italian Duke Gaetano Sforza. Unfortunately, his outstanding work has been completely forgotten till recently.

2. VOLUMES OF SPHERICAL OCTAHEDRA WITH SYMMETRIES

In the paper by N. V. Abrosimov, M. Godoy-Molina and A. D. Mednykh [13] the closed integral formulae for the volume of non-Euclidean octahedron and dual hexahedron having non-trivial symmetries are established. Trigonometrical identities involving lengths of edges and dihedral angles (Sine-Tangent Rules) are obtained. This gives a possibility to express the

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lengths in terms of angles. Then the Schläfli formula is applied to find the volume of polyhedra in terms of dihedral angles explicitly.

In particular, we consider octahedron $\mathcal{O} = \mathcal{O}(a, b, c, d, A, B, C, D)$ with edge lengthes a, b, c, d and corresponding dihedral angles A, B, C, D which has $2|m$ symmetry (Fig. 1).

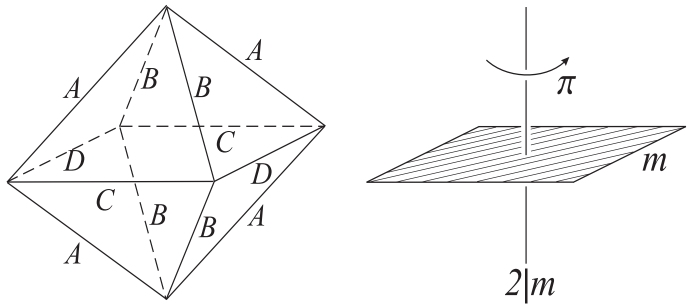

Fig. 1 Octahedron $\mathcal{O}(a, b, c, d, A, B, C, D)$ with $2|m$ symmetry

To find a volume of such octahedron in spherical space we obtain the following trigonometrical identity at first

Theorem 1. (The Sine-Tangent Rule) [13] *Let $\mathcal{O}(a, b, c, d, A, B, C, D)$ be a spherical octahedron with $2|m$ symmetry. Then the following trigonometric rule holds*

$$\frac{\sin^2 A}{\tan^2 a} = \frac{\sin^2 B}{\tan^2 b} = \frac{\sin^2 \frac{C+D}{2}}{\tan^2 \frac{c+d}{2}} = \frac{\sin^2 \frac{C-D}{2}}{\tan^2 \frac{c-d}{2}} = \tan^2 \theta$$

where $0 \leqslant \theta \leqslant \pi/2$ is a number defined by

$$\tan^2 \theta + \frac{(1+X)(1+Y)(1+Z)(1+W)}{X+Y+Z+W} = 0,$$

$$X = \cos A, Y = \cos B, Z = \cos \frac{C+D}{2} \text{ and } W = \cos \frac{C-D}{2}.$$

Thus, what we have realized is the important fact that a symmetric spherical octahedron is completely determined by its dihedral angles, hence $\mathcal{O} = \mathcal{O}(A, B, C, D)$. With this identity we are able to prove the following

Theorem 2. [13] *Let $\mathcal{O} = \mathcal{O}(A, B, C, D)$ be a spherical octahedron with $2|m$ symmetry. Then the volume of $\mathcal{O}$ is given by the formula*

$$V(\mathcal{O}) = \int_{\pi/2}^{\theta} \Big(\operatorname{arth}(X\cos\tau) + \operatorname{arth}(Y\cos\tau) + \operatorname{arth}(Z\cos\tau) + \operatorname{arth}(W\cos\tau)\Big)\frac{d\tau}{\cos\tau}$$

where $X = \cos A, Y = \cos B, Z = \cos \dfrac{C+D}{2}, W = \cos \dfrac{C-D}{2}$ and $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$ is a number defined by

$$\tan^2 \theta + \frac{(1+X)(1+Y)(1+Z)(1+W)}{X+Y+Z+W} = 0.$$

Analogous propositions like Theorem 1 and Theorem 2 take a place in hyperbolic case too.

3. Seidel problem on the volume of non-Euclidean tetrahedra

In 1986, Seidel conjectured that the volume of an ideal hyperbolic tetrahedron can be expressed as a function of the determinant and the permanent of its Gram matrix [14]. In spite of the fact that the volume of such a tetrahedron is known since Lobachevskii, the problem can not be solved for a long time. After 10 years, a strengthened version of this conjecture was stated by I. Rivin and F. Luo, who supposed that the volume of a tetrahedron (hyperbolic or spherical) depends only on the determinant of its Gram matrix.

In paper [15] N. V. Abrosimov show that the strengthened conjecture is false. Herewith, for spherical case the counterexample was constructed where the family of tetrahedra is given which continuously depends on free parameter varying in prescribed interval. All of those tetrahedra have the constant determinant of Gram matrix but their volume changes with the parameter. In the hyperbolic case, we have failed to construct an elementary counterexample to Seidel's strengthened conjecture. Nevertheless, a similar theorem is valid.

Theorem 3. [15] *There exists a one-parameter family of hyperbolic tetrahedra of different volumes whose Gram matrices have the same determinant.*

In the followed work [16] more strong result was obtained by author. The solution of Seidel's problem, which was posed in [14], is given by the following theorem.

Theorem 4. [16] *The volume of an ideal hyperbolic tetrahedron can be expressed as a function of the determinant and the permanent of its Gram matrix provided that this tetrahedron is either acute-angled or obtuse-angled*[1].

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[1]By an obtuse-angled tetrahedron we mean a tetrahedron in which at least one dihedral angle is obtuse.

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