COMBINATORIAL REALISATION OF CYCLES AND SIMPLICIAL VOLUME

ALEXANDER GAIFULLIN

The following problem on realisation of cycles was posed by Steenrod in 1940s. Given a homology class $z \in H_n(X, \mathbb{Z})$ of a topological space X, does there exist an oriented closed smooth manifold M^n and a continuous mapping $f : M^n \to X$ such that $f_*[M^n] = z$? If the answer is "yes", z is said to be realisable. In 1954, Thom found a non-realisable 7-dimensional class and proved that for every n, there is a positive integer $k(n)$ such that the class $k(n)z$ is always realisable.

We consider the following explicit version of Steenrod's problem. Given a singular cycle Z representing a homology class $z \in H_n(X, \mathbb{Z})$, can we construct *explicitly* a manifold M^n and a mapping $f : M^n \to X$ such that $f_*[M^n] = kz$ for a non-zero integer k? Such explicit construction was proposed by the author [1]. It is based on the explicit procedure for resolving singularities of the singular cycle Z. This construction has several applications.

First, for every n, it allows us to find a manifold M^n that has the following universality property (see [2]–[4]):

Universal Realisation of Cycles (URC) property. *For any X and any $z \in H_n(X, \mathbb{Z})$, a multiple of z can be realised by an image of some non-ramified finite-sheeted covering of M^n.*

This manifold M^n is a so-called small cover of the permutahedron, i.e., a manifold glued in a special way of 2^n permutahedra. (The permutahedron is a special convex polytope with $(n+1)!$ vertices.) Further, among small covers over other simple polytopes, we find a broad class of examples of URC-manifolds, i. e., manifolds that have URC-property. In particular, in dimension 4, we find a hyperbolic URC-manifold, thus proving a conjecture of Kotschick and Löh claiming that a multiple of any homology class can be realised by an image of a hyperbolic manifold.

Second application is to the theory of simplicial volume. Recall that the *simplicial semi-norm* on the homology of a topological space X is defined in the following way. Denote by $C_n(X, \mathbb{R})$ the n-dimensional singular simplicial chain group of X with real coefficients. For each chain $Z = \sum \alpha_i \sigma_i$, where $\alpha_i \in \mathbb{R}$ and σ_i are singular simplices, the ℓ^1-norm of it is equal to

$$\|Z\|_1 = \sum |\alpha_i|.$$

The simplicial semi-norm of a homology class $z \in H_n(X, \mathbb{R})$ is given by

$$\|z\|_1 = \inf \|Z\|_1,$$

where the infimum is taken over all singular cycles Z representing the homology class z. The simplicial semi-norm of the fundamental class $[M^n]$ of an oriented closed manifold M^n is called the *simplicial volume* (or the *Gromov norm*) of M^n, and is denoted by $\|M^n\|$.

It is well-known that in dimension 2 the simplicial semi-norm can be equivalently defined in the following way. For a homology class $z \in H_2(X, \mathbb{Z})$,

$$\|z\| = \inf \left\{ \frac{g}{k} \,\middle|\, f_*[S^2_g] = kz \right\},$$

where S^2_g is an oriented surface of genus g.

A natural question is whether it is possible to obtain a similar description of the simplicial semi-norm in an arbitrary dimension n. We have the following result towards this question.

The work was partially supported by RFBR (projects 12-01-31444, 12-01-92104), by a grant of the President of Russian Federation (project MD-4458.2012.1), by a grant of the Government of the Russian Federation (project 2010-220-01-077), and by Dmitri Zimin's "Dynasty" foundation.

For each URC-manifold M^n, we define the corresponding semi-norm $\|\cdot\|_{M^n}$ on homology in the following way. For any X and $z \in H_n(X, \mathbb{Z})$, we put

$$\|z\|_{M^n} = \inf \frac{r}{|k|},$$

where the infimum is taken over all mappings $f\colon \widehat{M^n} \to X$ such that $\widehat{M^n}$ is an r-sheeted covering of M^n and $f_*[\widehat{M^n}] = kz$, $k \neq 0$.

Theorem ([5]) *For each URC-manifold M^n, there exist positive constants $c_1(M^n)$ and $c_2(M^n)$ such that for any X and any $z \in H_n(X, \mathbb{Z})$, we have*

$$c_1(M^n)\|z\|_1 \leq \|z\|_{M^n} \leq c_2(M^n)\|z\|_1.$$

Indeed, one can take $c_1(M^n) = \|M^n\|^{-1}$. We do not know if $\|z\|_{M^n} = \|M^n\|^{-1} \cdot \|z\|_1$ for all z.

References

[1] A. A. Gaifullin, "Explicit construction of manifolds realising prescribed homology classes", *Russ. Math. Surveys*, 62, No. 6, 1199–1201 (2007), `arXiv:0712.1709`

[2] A. A. Gaifullin, "Realisation of cycles by aspherical manifolds", *Russ. Math. Surveys*, 63, No. 3, 562–564 (2008), `arXiv:0806.3580`

[3] A. A. Gaifullin, "The Manifold of Isospectral Symmetric Tridiagonal Matrices and Realization of Cycles by Aspherical Manifolds", *Proc. Steklov Inst. Math.*, 263, 38–56 (2008).

[4] A. A. Gaifullin, "Universal realisators for homology classes", `arXiv:1201.4823`, (2012).

[5] A. A. Gaifullin, "Combinatorial realisation of cycles and small covers", *Proc. of 6ECM*, to appear (2012), `arXiv:1204.4823`

Steklov Mathematical Institute, Moscow, 119991, Russia
Moscow State University, Moscow, 119991, Russia
Kharkevich Institute for Information Transmission Problems, Moscow, 127994, Russia
E-mail address: agaif@mi.ras.ru