MORSE-SARD THEOREM FOR SOBOLEV FUNCTIONS AND APPLICATIONS

MIKHAIL KOROBKOV

Theorem 1 [1]–[2]. *Let $\psi \in W^{n,1}(\mathbb{R}^n)$. Then*
(i) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $U \subset \mathbb{R}^n$ with $\mathcal{H}^1_\infty(U) < \delta$ the inequality $\mathcal{H}^1(\psi(U)) < \varepsilon$ holds;*
(ii) $\mathcal{H}^1(\{\psi(x) : x \in \mathbb{R}^n \ \& \ \nabla\psi(x) = 0\}) = 0$.

Here we denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure, i.e., $\mathcal{H}^1(F) = \lim\limits_{t\to 0+} \mathcal{H}^1_t(F)$, where $\mathcal{H}^1_t(F) = \inf\{\sum\limits_{i=1}^{\infty} \operatorname{diam} F_i : \operatorname{diam} F_i \le t, F \subset \bigcup\limits_{i=1}^{\infty} F_i\}$.

Corollary 2 [1]–[2]. *Let $\psi \in W^{n,1}(\mathbb{R}^n)$. Then for $\mathcal{H}^1$–almost all $y \in \psi(\mathbb{R}^n) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1–smooth $(n-1)$-dimensional compact manifolds S_j, $j = 1, 2, \ldots, N(y)$.*

Now consider the Euler system

$$\left\{\begin{array}{rcl} (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla p &=& 0, \\ \operatorname{div}\mathbf{w} &=& 0. \end{array}\right. \tag{1}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Assume that $\mathbf{w} = (w_1, w_2) \in W^{1,2}(\Omega, \mathbb{R}^2)$ and $p \in W^{1,s}(\Omega)$, $s \in [1, 2)$, satisfy the Euler equations (1) for almost all $x \in \Omega$ and let $\int_{\Gamma_i} \mathbf{w}\cdot\mathbf{n}\, dS = 0$, $i = 1, 2, \ldots, N$, where Γ_i are connected components of the boundary $\partial\Omega$. Then there exists a stream function $\psi \in W^{2,2}(\Omega)$ such that $\nabla\psi = (-w_2, w_1)$ (note that by Sobolev Embedding Theorem ψ is continuous in $\overline{\Omega}$) . Denote by $\Phi = p + \frac{|\mathbf{w}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{w}, p)$.

Theorem 3 [3] **(Bernoulli Law for Sobolev solutions).** *Under above conditions, for any connected set $K \subset \overline{\Omega}$ such that $\psi\big|_K = \mathrm{const}$ the assertion*

$$\exists C = C(K) \quad \Phi(x) = C \quad \text{for } \mathcal{H}^1\text{-almost all } x \in K$$

holds.

Using Theorem 3 we prove the existence of the solutions to steady Navier–Stokes equations for some plane cases (see [4]) and for the spatial case when the flow has an axis of symmetry (see [5]).

References

[1] J. Bourgain, M. V. Korobkov, J. Kristensen, "On the Morse–Sard property and level sets of $W^{n,1}$ Sobolev functions on $\mathbb{R}^n$", `http://arxiv.org/abs/1201.1416, [math.AP], 6 Jan 2012.`

[2] J. Bourgain, M. V. Korobkov, J. Kristensen, "On the Morse – Sard property and level sets of Sobolev and BV functions", `arXiv:1007.4408v1, [math.AP], 26 July 2010`, to appear in *Revista Mat. Iberoamericana.*

[3] M. V. Korobkov, "Bernoulli law under minimal smoothness assumptions", *Dokl. Math.*, 83, 107–110 (2011).

[4] M. V. Korobkov, K. Pileckas, R. Russo, "On the flux problem in the theory of steady Navier-Stokes equations with nonhomogeneous boundary conditions", `arXiv:1009.4024v2, [math-ph], 28 Oct 2011,` to appear in *Archive for Rational Mechanics and Analysis.*

[5] M. V. Korobkov, K. Pileckas, R. Russo, "Steady Navier-Stokes system with nonhomogeneous boundary conditions in the axially symmetric case", `arXiv:1110.6301v2, [math-ph], 11 Apr 2012.`

Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
E-mail address: korob@math.nsc.ru