

GEOMETRY OF SPACES OF RATIONAL FUNCTIONS

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Meromorphic functions are holomorphic mappings of complex curves to the projective line. Spaces of such functions on curves of given genus (*Hurwitz spaces*) are classical objects of investigation. One of the simplest examples of such spaces is the space of rational functions of degree 3 with simple poles considered up to rational changes of the domain variable. This space can be identified with the space of functions

$$az + \frac{b}{z} + \frac{c}{z-1} + d : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad a, b, c \neq 0,$$

that is, with an open subset in the vector space \mathbb{C}^4 with the coordinates a, b, c, d .

The last decades clarified the importance of Hurwitz spaces due to the fact that they are first instances of spaces of mappings of complex curves to algebraic varieties, which, in their own turn, are the main object of study in the theory of Gromov–Witten invariants.

Under the geometry of Hurwitz spaces we mean their cohomology properties and intersection theory on these spaces. The example above shows that Hurwitz spaces normally are noncompact, which makes necessary constructing their appropriate compactifications. The most well-known and widely used compactification, the one due to Harris and Mumford [2], possesses a disadvantage that it is singular. Another compactification was constructed in [1]. Both compactifications take into account mappings of singular curves, but in the second case only mappings to \mathbb{CP}^1 are considered, while the Harris–Mumford approach requires studying mappings to singular curves as well. As is shown in [5], the second compactification consists of *stable maps*, that is, maps whose automorphism group is finite.

Denote by $\overline{\mathcal{H}}_{g;n}$ the completed Hurwitz space of meromorphic functions on genus g curves with n marked poles of order 1. The example above corresponds to the case $g = 0$, $n = 3$, and the Hurwitz space $\overline{\mathcal{H}}_{0;3}$ is just \mathbb{C}^4 . In order to obtain a compact space, one has only to projectivize $\overline{\mathcal{H}}_{g;n}$, that is, to consider non-zero meromorphic functions up to a non-zero multiplicative constant.

Both the Hurwitz space $\overline{\mathcal{H}}_{g;n}$ and its projectivization $P\overline{\mathcal{H}}_{g;n}$ are naturally fibered over the moduli space $\overline{\mathcal{M}}_{g;n}$ of stable genus g complex curves with n marked points (the fibration is obtained by associating the domain to a function). As a result, there is no hope to get a complete description of the cohomology $H^*(\overline{\mathcal{H}}_{g;n})$ of the Hurwitz spaces, since the cohomology $H^*(\overline{\mathcal{M}}_{g;n})$ is known to be extremely complicated.

Fortunately, there is also no need too. Information required to do practical computations concerns the *degrees* of certain specific *strata* in a natural stratification of the Hurwitz spaces. The stratification in question of the Hurwitz spaces consists of functions having specific *singularities*. A generic function is a Morse function; it has singularities of type A_1 , that is, $z \mapsto z^2$ only. In addition, the space $P\overline{\mathcal{H}}_{0;3}$ contains the stratum of functions possessing the singularity A_2 , which is locally $z \mapsto z^3$.

The simplest and important numerical characteristic of a singularity stratum is its *degree* (the intersection index with the complementary power of the projectivization class $\mathcal{O}(1) \in H^2(P\overline{\mathcal{H}}_{g;n})$). The degrees of the strata coincide with the *Hurwitz numbers* thoroughly studied, in particular, by A. D. Mednykh, see e.g. [6]. These numbers are essentially the Gromov–Witten invariants of the projective line.

The main tool in the analysis of the geometry of the strata is the Thom–Kazaryan theory of universal polynomials, which allows one to express cohomology classes of the singularity

(and more general multi- and multimulti-singularity) strata in terms of few basic characteristic classes. Namely, the following statement is valid.

Theorem. [3, 4] *Any singularity (and multi- and multimulti-singularity) stratum in an arbitrary generic family of meromorphic functions on algebraic curves having only isolated singularities admits a universal expression in terms of four basic characteristic classes, which are relative Chern classes of the mappings of the family. The expression depends only on the singularity type, but is independent of the family under consideration.*

In particular, the universal formulas are applicable to more general Hurwitz spaces $P\overline{\mathcal{H}}_{g;m_1,\dots,m_n}$ that consist of meromorphic functions with marked poles of orders m_1, \dots, m_n .

In order to convert this statement into a tool for explicit computation of the degrees of the strata, that is, the Hurwitz numbers, one has to know not just the existence of the universal formulas, but the formulas themselves. Up to now they can be deduced in few important cases. In particular, the following statement is true.

Theorem (M. E. Kazaryan, S. K. Lando, 2012). *The generating function for the universal polynomials for the singularity strata in the presence of local singularities of types A_m only is a solution to the KP hierarchy of partial differential equations.*

The Kadomtsev–Petviashvili (KP) hierarchy is an important integrable system of partial differential equations originating in mathematical physics. Knowing that certain generating series is a solution to this hierarchy, provides recurrence relations for the coefficients of the series. The last statement seems to indicate the first instance of a solution to KP with cohomology coefficients.

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