RATIONAL TRIGONOMETRY OF A TETRAHEDRON

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This talk will outline a new approach to the trigonometry of a general tetrahedron in Euclidean space, by combining the main formulas of planar affine rational trigonometry ([5]) and planar projective trigonometry ([4]) (which is the basis for Universal Hyperbolic Geometry ([1],[2],[3])).

The main metrical notions in the affine case are quadrance between two points, spread between two lines or two planes, quadreas of triangular faces, the solid spread made by a tripod of three concurrent lines in three dimensional space, and the quadrume of the tetrahedron itself. These notions will be here reviewed.

Suppose we have a symmetric bilinear form $v \cdot w \equiv vMw^T$ between (row) vectors v and w, for a symmetric non-degenerate matrix M, in a vector space over a field, which may be taken to be the rational numbers, or even a finite field. The **quadrance** of the vector v is the number

$$Q(v) \equiv v \cdot v.$$

The **spread** between the vectors v and w is the number

$$s(v,w) \equiv 1 - \frac{(v \cdot w)^2}{Q(v)\,Q(w)}.$$

If P and R are two planes with normal vectors p and r respectively, then the **spread** $S(P,R)$ between the planes is

$$S(P,R) \equiv s(p,r).$$

If a triangle has side vectors v, w and u (so that say $v+w+u=0$) with respective quadrances Q_v, Q_u and Q_w, then the **quadrea** of the triangle is the number

$$\begin{aligned}\mathcal{A} &\equiv (Q_v + Q_u + Q_w)^2 - 2\left(Q_v^2 + Q_u^2 + Q_w^2\right) \\ &= 4Q_vQ_u - (Q_v + Q_u - Q_w)^2.\end{aligned}$$

In Euclidean geometry $\mathcal{A}$ is 16 times the square of the area of the triangle, this is a rational form of Heron's formula.

If v, w and u are three row vectors which are the directions of three concurrent lines forming a tripod, then the **solid spread** $\mathcal{S}$ of the tripod (also called a projective triangle) is

$$\mathcal{S} \equiv \frac{\det^2 \begin{pmatrix} v \\ u \\ w \end{pmatrix}}{Q(v)\,Q(u)\,Q(w)}.$$

So altogether a general tetrahedron has 6 edge quadrances (one for each edge), 12 face spreads (three for each face), 6 dihedral edge spreads (between faces which meet at an edge), 4 solid spreads (one at each vertex) and one quadrume $\mathcal{V}$, a multiple of the square of the volume.

Our aim is to introduce some of the many relations between these numbers. Some of these are obvious analogs of classical formulae known for some time, such as the formula for the volume of a tetrahedron in terms of the edge quadrances, going back to Tartaglia. Others of the formulas will perhaps not be so familiar! We will motivate our discussion by simple tetrahedra that can be constructed with the popular construction set Zome.

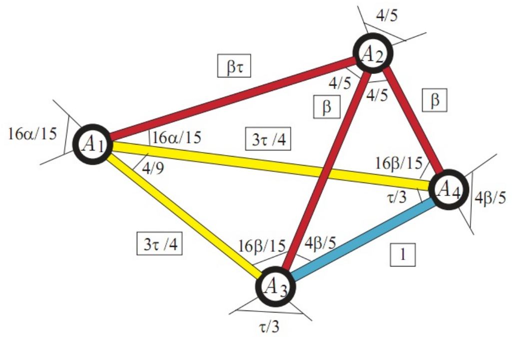

FIGURE 1. A Zome tetrahedron associated to the dodecahedron

It is of course interesting to contemplate the extension of such formulae to the spherical or hyperbolic cases.

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