Completely regular codes in the Johnson graphs $J(v, 3)$

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The *Johnson graph* $J(v, k)$ is a graph whose vertex set consists of all k-subsets of a fixed v-set; two k-subsets are adjacent if and only if they share $k-1$ elements exactly. Since the graphs $J(v, k)$ and $J(v, v-k)$ are isomorphic, we assume that $k \leq v/2$. The Johnson graph is distance-regular, see [1], has diameter k and $k+1$ distinct eigenvalues $\theta_i = (k-i)(v-k-i)-i$, $i = 0, \ldots, k$.

A *perfect coloring* with r colors of a graph Γ (r-coloring, for short) is a partition of the vertex set of Γ into r parts (colors) $C_1, \ldots, C_r$ such that, for all $i, j \in \{1, \ldots, r\}$, every vertex from C_i is adjacent to the same number of vertices, namely, c_{ij}, from C_j. The matrix $C = (c_{ij})_{i,j=1,\ldots,r}$ is called the *quotient matrix* of a perfect r-coloring. It is well known [1] that the eigenvalues of the quotient matrix are eigenvalues of the graph.

Given a perfect r-coloring, if an appropriate ordering of its colors determines a distance partition of the vertex set with respect to C_i, then C_i is called a *completely regular code*. In this case the number $\rho := r - 1$ is called the *covering radius* of code. The set of vertices at maximal distance from a completely regular code also is a completely regular code and is called the *opposite code* [4].

A *t-design* is a set $\mathcal{B}$ of vertices of $J(v, k)$ (usually called blocks) such that every t–subset of v–set belongs to the constant number of blocks from $\mathcal{B}$ The *strength* of $\mathcal{B}$ is the largest t such that $\mathcal{B}$ is a t-design.

One may show [4] that the strength of a completely regular code of $J(v, k)$ is t such that θ_{t+1} is the second largest eigenvalue of the quotient matrix.

Completely regular codes of strength 0 in the Hamming graphs and Johnson graphs have been described in [3]. Completely regular codes of strength 1 with assumption that a code or its opposite induces a disconnected subgraph in $J(v, k)$ have been described in [4]. Completely regular codes with covering radius 1 in $J(v, k)$ for $v \leq 8$ or $k = 2$ are studied in [2], [5], [6].

Given a completely regular code in $J(v, k)$, the inequality $t + \rho \leq k$ holds [1], where t is its strength, and ρ is its covering radius. We also recall, see [2], that any $(k-1)$-design is a completely regular code in $J(v, k)$. Hence, for $k = 3$, the parameters $(t, \rho) \in \{(1, 1), (1, 2)\}$ remain unsolved.

Completely regular codes in $J(v,3)$ with $(t,\rho) = (1,1)$ and under assumption that the quotient matrix is symmetric or v is even have been classified in [7].

In the present work, we finish the classification of completely regular codes in $J(v,3)$.

Теорема. *The quotient matrix of a perfect* 2*-coloring that corresponds to a completely regular code of strength* 1 *and radius* 1 *in* $J(v,3)$ *is one of the following*:

$$\begin{pmatrix} 3(v-5) & 6 \\ 4(\frac{v}{2}-2) & v-1 \end{pmatrix}, \begin{pmatrix} 3(\frac{v}{2}-3) & \frac{3v}{2} \\ \frac{v}{2}-2 & \frac{5v}{2}-7 \end{pmatrix}, \begin{pmatrix} 3(\frac{v}{2}-1) & 3(\frac{v}{2}-2) \\ \frac{v}{2}+4 & \frac{5v}{2}-13 \end{pmatrix},$$

in particular, v is even. Conversely, for any of the matrices above, there is a perfect 2*-coloring of* $J(v,3)$ *with corresponding quotient matrix.*

Conclusion. Recently, there have been several results on the perfect colorings of various graphs, utilizing the local structure of the graphs [5], [7], [8]. In this work, using the approach, we obtain the classification of completely regular codes in $J(v,3)$ with covering radius $\rho = 1$ of strength 1. Moreover, we announce that the case when $\rho = 2$ in $J(v,3)$ can be also solved using the approach.

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