Direct product of locally primitive Cayley graphs

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Abstract

We determine some conditions in which the Cayley graph of product of two groups in which the Cayley graph of them are locally primitive, is locally primitive

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1 Introduction

Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of Γ. An edge joining the vertices u and v is denoted by $\{u, v\}$. The group of automorphisms of Γ is dnoted by $Aut(\Gamma)$, which acts on vertices, arcs and edges of Γ. Γ is called vertex transitive, edge transitive or arc transitive if $Aut(\Gamma)$ acts transitively on the set of vertices, edges or arcs of Γ respectively. If Γ is vertex and edge transitive, but not arc transitive, then it is called $\frac{1}{2}$-arc transitive.Γ is called X-locally primitive for $X \leqslant Aut(\Gamma)$ (or simply locally primitive when $X = Aut(\Gamma)$) if X_v acts primitively on $\Gamma(v)$ for each vertex $v \in V(\Gamma)$, where $\Gamma(v)$ is the set of vertices which are adjacent to v. It is known that 2-arc-transitive graphs form a proper subclass of vertex transitive locally primitive graphs.

Let G be a finite group and S be an inverse closed subset of G, i.e. $S = S^{-1}$, such that $1 \notin S$. The Cayley graph $\Gamma = Cay(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} | g \in G, s \in S\}$. Γ is connected if and only if $< S >= G$.For $g \in G$ define the mapping $\rho_g : G \longrightarrow G$ by $\rho_g(x) = xg, x \in G$. $\rho_g \in Aut(G)$ for every $g \in G$, thus $R(G) = \{\rho_g | g \in G\}$ is a regular subgroup of $Aut(\Gamma)$ isomorphic to G, forcing Γ to be a vertex transitive graph.

Let $\Gamma = Cay(G,S)$ be a Cayley graph of a finite group G on S. Let $Aut(G,S) = \{\alpha \in Aut(G) \mid S^\alpha = S\}$ and $A = Aut(\Gamma)$. Then the normalizer of $R(G)$ in A is equal to

$$N_A(R(G)) = R(G) \rtimes Aut(G,S)$$

where $\rtimes$ denotes the semi-direct product of two groups. In [10] the graph Γ is called normal if $R(G)$ is a normal subgroup of $Aut(\Gamma)$.

Therefore according to [2], $\Gamma = Cay(G,S)$ is normal if and only if $A := Aut(\Gamma) = R(G) \rtimes Aut(G,S)$, and in this case $A_1 = Aut(G,S)$ where A_1 is the stabilizer of the identity element of G under A. The normality of Cayley graphs has been extensively studied from different points of views by many authors. In [9] all disconnected normal Cayley graphs are obtained. Therefore, it suffices to study the connected Cayley graphs when one investigates the normality of Cayley graphs, which we use in this document.

Therefore in this document when we talk about a Cayley graph Γ, we mean $\Gamma = Cay(G,S)$, where G is a finite group and S is a non-empty generating subset of G such that $1 \notin S$ and $S = S^{-1}$. We also denote $Aut(\Gamma)$ by A.

Locally primitive Cayley graphs and 2-arc-transitive Cayley graphs have been exten-sively studied, see for example,[1], [5], [6], [7] , [8] and references therein. Also, normal Cayley graphs have received much attention in the literature, see for example, [3] and [4]. In particular, 2-arc-transitive normal Cayley graphs of elementary abelian groups are classified by Ivanov and Praeger [3]. This motivates the author to find some conditions on which the direct product of locally primitive normal Cayley grpahs are locally primitive as an expansion of the locally primitive Cayley graphs.

2 Main Results

Theorem 2.1 *Let H and K be two groups which have no common direct factor and satisfy the condition $gcd(|H/H'|, |Z(K)|) = 1 = gcd(|K/K'|, |Z(H)|)$, S an inverse closed subset of H and T an inverse closed subset of K and non of them have the identity.If $\Gamma_1 = Cay(H,S)$ is normal and X-locally primitive $\Gamma_2 = Cay(K,T)$ is normal and Y-locally primitive Cayley graph, then $\Gamma = Cay(H\times K, S\times T)$ is also $X\times Y$ locally primitive Cayley graph.*

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