**Kondratiev A. S., Loginov S. M.**

*e-mail: a.s.kondratiev@imm.uran.ru, smloginov@gmail.com*

**Finite almost simple 5-primary groups and their Gruenberg-Kegel graphs**

Let $G$ be a finite group. Denote by $\pi(G)$ the set of prime divisors of the order of $G$. If $|\pi(G)| = n$ then the group $G$ is called $n$-primary. The prime graph (Gruenberg-Kegel graph) $\Gamma(G)$ of $G$ is defined as a graph with the vertex set $\pi(G)$, in which two different vertices $p$ and $q$ are adjacent if and only if there exists an element of order $pq$ in $G$.

Many investigators in the finite group theory are interested by various particular cases of the general problem of the study of finite groups by the properties of their Gruenberg-Kegel graphs. In the frame of this general problem, our attention draw first of all a more detailed study of the class of finite groups with disconnected prime graph. In fact, this class generalizes widely the class of finite Frobenius groups as is obvious from the well-known stuctural Gruenberg-Kegel theorem on finite groups with disconnected prime graph (see [8]). And Frobenius groups occupy an absolutely exceptional place in the finite group theory. Note also that the class of finite groups with disconnected prime graph coincides with the class of finite groups having an isolated subgroup (i. e., a proper subgroup containing the centralizer of any its nontrivial element) which have studied by many known algebraists (Frobenius, Suzuki, Feit, Thompson, G. Higman, Arad, Chillag, Busarkin, Gorchakov, Podufalov and others).

Finite simple groups with disconnected prime graph are determined in the papers of Williams [8] and the first author [3]. They compose sufficiently restricted class of all finite simple groups, but include many "small" in various senses groups which arise often in the investigations. For example, all finite simple groups of exceptional Lie type, besides the groups $E_7(q)$ for $q > 3$, and also all finite simple groups from well-known "Atlas of finite groups" [1], besides the group $A_{10}$, have disconnected prime graph. The classification of connected components of prime graph for finite simple groups, obtained in [8, 3], were applied by Lucido [7] for the obtaining an analogous classification for all finite almost simple groups, i. e., groups with nonabelian simple socle. The problem of the study of finite groups with disconnected prime graph, which are not almost simple, is solved for several particular cases only, because here some nontrivial problems related with modular representations of finite almost simple groups arise.

In the frame of above-mentioned problem, the first author and Khramtsov in [4, 5, 6] studied finite groups having disconnected prime graph whose the number of vertices is at the most 4. We continue these investigations with the purpose of the study of finite 5-primary groups with disconnected prime graph. In the given work, we make a necessary preliminary step for this by the determining the finite almost simple 5-primary groups and their Gruenberg-Kegel graphs. In addition, a list of finite simple 5-primary groups obtained in [2, 9] is essentially

refined. In particular, the following theorem having an independent interest is proved.

**Theorem.** *A finite almost simple group $G$ is 5-primary if and only if one of the following statements holds:*

(1) *the socle of $G$ is isomorphic to one of the groups $A_{11}$, $A_{12}$, $L_2(q)$ for $q \in \{2^8, 2^9, 5^3, 5^4, 7^3, 7^4, 7^7, 17^2, 17^3\}$, $L_3(9)$, $L_3(27)$, $L_4(q)$ for $q \in \{4, 5, 7\}$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $U_3(q)$ for $q \in \{16, 17, 25, 81\}$, $U_4(q)$ for $q \in \{4, 5, 7, 9\}$, $U_5(3)$, $U_6(2)$, $S_4(q)$ for $q \in \{8, 16, 17, 25, 49\}$, $S_6(3)$, $S_8(2)$, $O_7(3)$, $O_8^+(3)$, $O_8^-(2)$, $G_2(q)$ for $q \in \{4, 5, 7, 8\}$, $M_{22}$, $J_3$, $HS$, $He$ or $M^cL$;*

(2) *$G \cong L_2(2^p)$, where $p$ is a prime, $p \geq 11$ and $|\pi(2^{2p} - 1)| = 4$;*

(3) *$G \cong Aut(L_2(2^p))$, where $p$, $2^p - 1$ and $(2^p + 1)/3$ are some different primes and $p \geq 7$;*

(4) *$G \cong L_2(p)$ or $PGL_2(p)$, where $p$ is a prime, $p \geq 41$ and $|\pi(p^2 - 1)| = 4$;*

(5) *the socle of $G$ is isomorphic to $L_2(p^2)$, where $p$ is a prime, $p \geq 11$, $|\pi(p^2 - 1)| = 3$ and $p^2 + 1 = 2r$ or $2r^2$ for an odd prime $r$;*

(6) *$G$ is isomorphic to $L_2(p^r)$ or $PGL_2(p^r)$, where $p \in \{3, 5, 7, 17\}$, $r$ is a prime, $r$ does not divide $p(p^{2r} - 1)$ and $|\pi(p^{2r} - 1)| = 4$;*

(7) *$G \cong Aut(L_2(3^p))$ or $O^2(Aut(L_2(3^p)))$, where $p$ and $(3^p-1)/2$ are primes, $p \geq 5$, $|\pi(3^p + 1)| = 2$ and $p$ does not divide $3^{2p} - 1$;*

(8) *$G \cong L_3^\epsilon(3^p)$ or $O^p(Aut(L_3(3^p)))$, where $\epsilon \in \{+, -\}$, $p$ and $(3^p - 1)/2$ are primes, $p \geq 5$, $|\pi(3^p + 1)| = 2$, $p$ does not divide $3^{2p} - 1$ and $|\pi(\frac{3^{2p}+\epsilon 3^p+1}{(3,p-\epsilon 1)})| = 1$;*

(9) *the socle of $G$ is isomorphic to $L_3^\epsilon(p)$, where $\epsilon \in \{+, -\}$, $p$ is a prime, $17 \neq p \geq 11$, $|\pi(p^2 - 1)| = 3$, and $|\pi(\frac{p^2+\epsilon p+1}{(3,p-\epsilon 1)})| = 1$;*

(10) *$G \cong S_4(p)$ or $PGSp_4(p)$, where $p$ is a prime, $p \geq 11$, $|\pi(p^2 - 1)| = 3$ and $p^2 + 1) = 2r$ or $2r^2$ for an odd prime $r$;*

(11) *$G \cong Sz(2^p)$, where $p$ and $2^p - 1$ are primes, $p \geq 7$, $|\pi(2^{2p} + 1)| = 3$;*

(12) *$G \cong Aut(Sz(8))$.*

The results of this paper show that finite simple 5-primary groups besides the groups $L_4(q)$ for $q \in \{4, 7\}$ and $U_4(q)$ for $q \in \{4, 5, 7, 9\}$ have disconnected prime graph.

We use the notation from [1].

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