SUBEXTENSIONS FOR A PERMUTATION $\mathrm{PSL}_2(q)$-MODULE

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We denote by $\mathbb{F}_q$ a finite field of order q and by $\mathbb{Z}_n$ a cyclic group of order n.

Let q be an odd prime power and let $G = \mathrm{PSL}_2(q)$. From the Universal Embedding Theorem [1, Theorem 2.6.A], it follows that the regular wreath product $\mathbb{Z}_2 \wr G$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$. It is of interest to know if the same is true for a permutation wreath product that is not necessarily regular. In particular, let ρ be the natural permutation representation of G of degree $q+1$ on the projective line over $\mathbb{F}_q$. The following problem arose in the research [2].

Problem 1. *Does the permutation wreath product $\mathbb{Z}_2 \wr_\rho G$ contain a subgroup isomorphic to $\mathrm{SL}_2(q)$?*

Although stated in purely group-theoretic terms, this problem is cohomological in nature. We reformulate a generalized version of this question as an assertion about a homomorphism between second cohomology groups of group modules. We then apply some basic cohomology theory to obtain the following.

Theorem 1. *If $q \equiv -1 \pmod 4$ then the answer to Problem 1 is affirmative.*

It seems that the case $q \equiv 1 \pmod 4$ is more complicated and requires some deeper considerations than those presented here. We put forward

Conjecture 1. *If $q \equiv 1 \pmod 4$ then $\mathrm{SL}_2(q)$ is not embedded in $\mathbb{Z}_2 \wr_\rho G$.*

For small values $q = 5, 9, 13, 17$, Conjecture 1 was confirmed using a computer. We show how to reduce this problem to the determination of the first cohomology groups $H^1(G, U_\pm)$, where U_+ and U_- are the two nontrivial absolutely irreducible G-modules in the principal 2-block.

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References

[1] J. D. Dixon, B. Mortimer, Permutation groups. Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996, 346 p.

[2] N. V. Maslova, D. O. Revin, On composition factors of a finite group that is minimal with respect to the prime spectrum. (in preparation)