ON FINITE ALPERIN GROUPS WITH ABELIAN SECOND COMMUTATOR SUBGROUPS

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J.L. Alperin studied in [1] groups in which all 2-generated subgroups have a cyclic commutator subgroup. Such groups are called Alperin groups. It was proved in [1] that, for odd prime p , finite Alperin p-groups are metabelian; i.e., they have an abelian commutator subgroup. However, finite Alperin 2-groups may be nonmetabelian. For example, a nonmetabelian finite Alperin 2-group with second commutator subgroup of order 2 was constructed in [2], and infinite series of finite Alperin 2-groups with second commutator subgroups of orders 2 and 4 were constructed in [3].

In [4,5] the author proved the existence of finite Alperin 2-groups with cyclic second commutator subgroup of arbitrarily large order and with elementary abelian second commutator subgroup of arbitrary rank.

In this communication we announce for every finite abelian group H the existence of a finite Alperin group G whose second commutator subgroup G'' is isomorphic to H. This result is an easy consequence of the following theorem.

Below a homocyclic group means a group which is isomophic to $Z_m \times \cdots \times Z_m$ where m is a positive integer.

Theorem 1 *Let m, n be positive integers, $n \geq 3$ and let a group G be defined by generators a_i, f_{ij}, τ_{ijk}, where $1 \leq i, j, k \leq n$, and defining relations:*

1) $a_i^2 = 1$,

2) $[a_i, a_j] = f_{ij}$,

3) $[f_{ij}, a_k] = f_{jk}^2 f_{ki}^2 \tau_{ijk}^{-2}$,

4) $[\tau_{ijk}, a_s] = 1$,

5) $f_{ij}^{4m} = 1$,

6) $\tau_{ijk}^m = 1$,

7) $[f_{ij}, f_{ks}] = \tau_{kjs}\tau_{ksi}$,

8) $(f_{ij}f_{jk}f_{ki})^4 = \tau_{ijk}$,

9) $\tau_{sij}\tau_{sjk}\tau_{ski} = \tau_{ijk}$,

for all positive integers $i, j, k, s \in [1, n]$.

Then the following statements are valid:

I) The group G has a normal series:

$$1 < \langle \tau_{ijk} | 1 \leq i, j, k \leq n \rangle = G'' < G''\langle f_{12}\rangle < G''\langle f_{12}\rangle\langle f_{13}\rangle < \cdots <$$

$$G''\langle f_{12}\rangle\langle f_{13}\rangle \ldots \langle f_{1n}\rangle = H < H\langle f_{23}\rangle < H\langle f_{23}\rangle\langle f_{24}\rangle < \cdots <$$

$$H\langle f_{23}\rangle\langle f_{24}\rangle \ldots \langle f_{n-1,n}\rangle = G' < G'\langle a_1\rangle < G'\langle a_1\rangle\langle a_2\rangle < \cdots <$$

$$G'\langle a_1\rangle\langle a_2\rangle \ldots \langle a_n\rangle = G,$$

in which the first factor-group has order $m^{\binom{n-1}{2}}$, the following $(n-1)$ factor-groups are cyclic of order $4m$, then the following $\binom{n-1}{2}$ factor-groups are cyclic of order 4, and the last n factor-groups are cyclic of order 2;

II) $d(G) = n$, $G = \langle a_1, \dots, a_n \rangle$, $|a_i| = 2$ for any $i \in [1, n]$, $|G| = m^{\binom{n}{2}} 2^{n^2}$, $d(G') = \binom{n}{2}$, $G' = \langle f_{ij} | 1 \le i < j \le n \rangle$, $G'' = \prod_{2 \le i < j \le n} \langle \tau_{1ij} \rangle$ – homocyclic group with period m and rank $\binom{n-1}{2}$, $G'' \le Z(G)$;

III) G is an Alperin group, moreover for any $x, y \in G$ at least one of the following equalities is true: $[x, y, y] = 1$ or $[x, y, y] = [x, y]^{-2}$.

The following theorem is easily deduced from theorem 1.

Theorem 2 *For any finite abelian group H there exists a finite Alperin group G, generated by involutions, whose second commutator subgroup G'' is isomorphic to H.*

References

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