Periodic groups acting freely on Abelian groups
Lytkina D.. V., Mazurov V. D., Sozutov A I., Zhurtov A. Kh.

Let π be a set of primes. Periodic group G is called a *π-group*, if all of its element orders are divisible only by primes from π. A group G *acts freely* on a non-trivial group V, and the action is called a *free action*, if $vg = v$ implies $v = 1$ or $g = 1$ for $v \in V$ and $g \in G$.

Our goal is to describe $\{2,3\}$-groups acting freely on Abelian groups.

The interest to this topic was raised by a work of E. Jabara and P. Mayr [1], who proved that a $\{2,3\}$-group G of finite period 2^m3^2, acting freely on Abelian group is locally finite. Later D. V. Lytkina showed that this result is also true when we do not assume that a period of a Sylow 2-subgroup of G is finite.

Locally cyclic group G is a group such that every finite subset of G is contained in a cyclic subgroup.

Quaternion group is a quaternion group of order 8 or generalized quaternion group, i. e. a group isomorphic to a group

$$Q_{2^{m+1}} = \langle a, b \mid a^{2^m} = b^4 = 1,\ a^b = a^{-1},\ b^2 = a^{2^{m-1}} \rangle, \quad m \geqslant 3.$$

Locally quaternion group is a 2-group G such that every finite subset of G is contained in a quaternion subgroup.

Quaternion group of order 8 possesses an automorphism of order 3. The corresponding semidirect product of order 24 is isomorphic to the group $SL_2(3)$. Denote by $\tilde{S}_4$ an extension of a group of order 2 with a group S_4 of degree 4, which has a quaternion Sylow 2-subgroup.

A group G is called a *central product* of its subgroups A and B with the union by a subgroup C, if $G = AB$, $[A, B] = 1$ and $C = A \cap B$. Central product AB is said to be *non-trivial*, iff $A \neq G \neq B$.

Let p be an odd prime and a be a positive integer. We say that an infinite p-group P is a *group of the type* $Q(p^a)$ (or a $Q(p^a)$-type group), if P possesses the following properties:

1. $Z(P)$ is a cyclic group of order p^a.
2. Every finite subgroup of P is cyclic.

Note, that groups of the type $Q(p^a)$ are not locally finite.

We say that a $\{2,p\}$-group H with an involution is a *group of the type* $Q(p^a, d)$ if H possesses the following properties:

1. $Z(H)$ is locally cyclic.
2. Every 2-element of H belongs to $Z(H)$; all 2-elements form a group T of order 2^d (it is possible that $d = \infty$).
3. H/T is a group of the type $Q(p^a)$ for some positive integer a.

Theorem 1. *Let G be a $\{2,3\}$-group acting freely on an Abelian group. Then one of the following statements is true.*

1) *G is locally finite and isomorphic to one of the following groups*:

- *locally cyclic group*;
- *direct product of a locally cyclic* 3*-group with a locally quaternion group*;
- *semidirect product of a locally cyclic* 3*-group R with a cyclic* 2*-group $\langle b\rangle$, where $b^2 \neq 1$ and $a^b = a^{-1}$ for every $a \in R$*;
- *semidirect product of a locally cyclic* 3*-group R with a locally quaternion group Q, where $|Q : C_Q(R)| = 2$*;
- *semidirect product of a quaternion group $Q_8 = \langle x, y\rangle$ of order* 8 *with a cyclic* 3*-group $\langle a\rangle$, where $x^a = y$*;
- *group $\tilde{S}_4$.*

2) *G is not locally finite and all prime order elements of G generate a cyclic subgroup.*

Any of the groups mentioned can act frelly on some Abelian group.

We fully describe groups from i. 2 of Theorem 1 in Theorem 2.

Theorem 2. *Let G be a non locally finite $\{2,p\}$-group, where p is an odd prime. All prime order elements of G generate a cyclic subgroup iff all* 2*-elements of G generate a* 2*-subgroup S, which is locally cyclic or locally quaternion. Besides, one of the following conditions is true*:

1) *$G = P \times S$, where P is a $Q(p^a)$-type group*;

2) *S is a non-trivial locally cyclic group, and G is $Q(p^a, d)$-type*;

3) *S is a locally cyclic group, and G is a non-trivial central product of S and a $Q(p^a, d)$-type group with the union by a subgroup of order 2^d*;

4) *S is a locally quaternion group, and G is a central product of S and a $Q(p^a, 1)$-type group with the union by a subgroup of order* 2;

5) *$S = Q_8$, $p = 3$, $|G : C_G(S)| = 3$ and G/S is a $Q(p^a)$-type group. Here $C_G(S)$ is either a $Q(p^a, 1)$-group, or a central product of a $Q(p^a)$-type group and a group of order* 2.

The work is supported by Russian Foundation of Basic Research (Grants 12–01–90006, 13-01-00505, 11–01–00456), the Federal Target Program (Contract No. 14.740.11.0346), and the Integration Project of the Siberian Division of the Russian Academy of Sciences for 2012–2014 (No. 14).

References

[1] **Jabara E., Mayr P.** Frobenius complements of exponent dividing $2^m \cdot 9$ // Forum Mathematicum. 2009. Vol. 21, No.1. P. 217–220.

[2] **Lytkina D.V.** Periodic groups acting freely on abelian groups // Algebra and Logica. 2010. V. 49, No.3. P. 256–264.