Primitive and measure-preserving system of elements on the varieties of metabelian and metabelian profinite groups

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Consider an ordered set (system) of elements $\{v_1, \ldots, v_l\}$, $1 \le l \le r$, in the free group F_r of rank r. Let G be a finite group. Define the verbal mapping $\varphi_{\{v_1,\ldots,v_l\}}$ from G^r into G^l by assigning to each $\overline{g} = (g_1, \ldots, g_n) \in G^r$ the element $(v_1(g_1, \ldots, g_r), \ldots, v_l(g_1, \ldots, g_r))$ in G^l. A system of elements $\{v_1, \ldots, v_l\}$ *preserves measure on* G if every $\overline{g} \in G^l$ appears as an image under $\varphi_{\{v_1,\ldots,v_l\}}$ with probability $|G|^{-l}$. A system of elements $\{v_1, \ldots, v_l\}$, $1 \le l \le r$, that preserves measure on every finite group G is called *measure-preserving.*

A system of elements $\{v_1, \ldots, v_l\}$, $1 \le l \le r$, in F_r is called *primitive* if it can be complemented to a basis for F_r.

The following conjecture by several authors about the connection between primitive elements and measure-preserving elements was formulated:

Conjecture 1. *A system of elements* $\{v_1, \ldots, v_l\}$, $1 \le l \le r$, *in a free group* F_r *is primitive if and only if the system preserves measure.*

The conjecture was confirmed for $l \ge r - 1$ and late, for $l = 1$.

Suppose that as finite groups G we consider only the groups in some variety $\mathfrak{M}$. Let $V = V(\mathfrak{M})$ be the verbal subgroup in F_r corresponding to this variety. As $\{v_1, \ldots, v_l\}$ in the definition of systems of measure-preserving elements, we may consider elements in the free group $F_r(\mathfrak{M}) = F_r/V$ of $\mathfrak{M}$.

Replacing in the definition of systems of elements an arbitrary finite group G with an arbitrary finite group in $\mathfrak{M}$, and a free group F_r, with a relatively free group $F_r(\mathfrak{M})$, we arrive at the notation of systems of elements in $F_r(\mathfrak{M})$ that *preserve measure on* $\mathfrak{M}$.

By analogy with the definition of system of primitive elements in F_r, we can define *primitive systems of elements* in $F_r(\mathfrak{M})$ as system that can be included in some basis of $F_r(\mathfrak{M})$.

The above-formulated conjecture can be expressed for a group variety $\mathfrak{M}$:

Conjecture 2. *A system of elements* $\{v_1, \ldots, v_l\}$, $1 \le l \le r$, *in the free group* $F_r(\mathfrak{M})$ *is primitive if and only if the system preserves measure on* $\mathfrak{M}$.

We use the primitivity criterions for varieties of metabelian groups for proof the next theorems.

THEOREM 1. *Let* S *be a free metabelian group of rank* $r \ge 2$. *An element* v *preserves measure on the variety* $\mathfrak{A}^2$ *of all metabelian groups if and only if* v *is primitive.*

THEOREM 2. *Let* S *be a free metabelian group of rank* $r \ge 2$. *A system of elements* $\{v_1, \ldots, v_r\}$ *preserve measure on the variety* $\mathfrak{A}^2$ *of all metabelian groups if and only if they form a basis for* S.

THEOREM 3. *A system of elements* $\{v_1, \ldots, v_l\}, 1 \le l \le r$, *in a free profinite* $\mathfrak{A}^2$ *- group* $\widehat{S}_r$ *is primitive if and only if this system preserves measure on the variety of profinite* $\mathfrak{A}^2$ *- groups.*

THEOREM 4. *Suppose that* v *belongs to a free metabelian group* S_r *and* $\widehat{S}_r$ *is the profinite completion of* S_r. *The element* v *is primitive in* S_r *if and only if* v *is primitive in* $\widehat{S}_r$.

THEOREM 5. *Let elements* $\{v_1, \ldots, v_r\}$ *be chosen in a free metabelian group* S_r. *They constitute a basis for* S_r *if and only if they are a basis for profinite completion* $\widehat{S}_r$ *of* S_r.

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