Irreducible representations of the classical algebraic groups with p-large highest weights: properties of unipotent elements and restrictions to subgroups

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Irreducible representations of the classical algebraic groups in positive characteristic with large highest weights with respect to the characteristic are considered. We investigate special properties of such representations concerned with the behaviour of unipotent elements and restrictions to subsystem subgroups. For the classical algebraic groups in characteristic $p > 0$ the notion of a p-large representation was introduced in [2] in order to distinquish some regularities that are specific for modular representations, do not depend upon a fixed characteristic, and hold when the highest weight is large enough with respect to the characteristic. For a dominant weight $\mu = \sum_{j=0}^{t} p^j \lambda_j$ with p-restricted λ_j set $\overline{\mu} = \sum_{j=0}^{t} \lambda_j$ (this weight is uniquely determined). We call an irreducible representation of a simple algebraic group in characteristic $p > 0$ with highest weight ω p-large if the value of the weight $\overline{\omega}$ on the maximal root of the group is at least p.

I. Properties of unipotent elements in p-large representations. Lower estimates for the number of Jordan blocks of size p in the images of elements of order p in such representations in terms of the highest weight coefficients and the group rank are obtained. This allows one to get estimates for coranks of the images of arbitrary unipotent elements in relevant representations.

In what follows K is an algebraically closed field of characteristic p, G is a simply connected algebraic group of a classical type over K, r is the rank of G, ω_i and α_i, $1 \leq i \leq r$, are its fundamental weights and simple roots, and $\omega(\varphi)$ is the highest weight of an irreducible representation φ. For an irreducible representation φ denote by $l(\varphi)$ the value of $\omega(\varphi)$ on the maximal root. We assume that $p \neq 2$ for $G \neq A_r(K)$. If $\omega(\varphi) = \sum_{i=1}^{r} a_i\omega_i$, it is well known that

$$l(\varphi) = \begin{cases} \sum_{i=1}^{r} a_i & \text{for } G = A_r(K) \text{ or } C_r(K), \\ a_1 + 2(a_2 + \ldots + a_{r-1}) + a_r & \text{for } G = B_r(K), \\ a_1 + 2(a_2 + \ldots + a_{r-2}) + a_{r-1} + a_r & \text{for } G = D_r(K). \end{cases}$$

In Theorems 1–3 and Corollary 4 φ is a p-restricted irreducible representation of G with highest weight $\omega = \sum_{i=1}^{r} a_i\omega_i$. For $G = A_r(K)$ set $\omega^* = \sum_{i=1}^{r} a_{r+1-i}\omega_i$ (this is the highest weight of the representation dual to φ).

Theorem 1. *Let $G = A_r(K)$, $r > 8$, and $l(\varphi) \geq p$.*

1). *Assume that $l(\varphi) \geq p+2$, or $\sum_{i=3}^{r-2} a_i \neq 0$, or $a_2+a_{r-1} > 1$, or $a_2+a_{r-1} \neq 0$ and $l(\varphi) > p$. Then for a unipotent element $x \in G$ of order p the image $\varphi(x)$ has at least $(r-2)^3/8$ Jordan blocks of size p.*

2). *Set $\Omega = \{(p-1)\omega_1 + \omega_2 + \omega_r, (p-2)\omega_1 + 2\omega_2, a_1\omega_1 + a_r\omega_r, a_1 + a_r = p+2\}$. If ω satisfies the assumptions of Item 1, ω and $\omega^* \notin \Omega$, and, furthermore, both ω and $\omega^* \neq 2\omega_1 + \omega_3$ for $p = 3$, then $\varphi(x)$ has at least $(l(\varphi) - p + 2)(r-2)^3/8$ such blocks.*

Theorem 2. *Let $p > 2$, $G = C_r(K)$, $r > 12$, and $l(\varphi) \geq p$. Assume that $\omega \neq (p-1)\omega_1 + \omega_2$. Then for a unipotent element $x \in G$ of order p the image $\varphi(x)$ has at least $(r-1)^3$ Jordan blocks of size p. If $\omega \neq (p-2)\omega_1 + 2\omega_2$ and for $p = 3$ the weight $\omega \neq 2\omega_1 + \omega_3$ as well, then $\varphi(x)$ has at least $(l(\varphi) - p + 2)(r-1)^3$ such blocks.*

Theorem 3. *Let $p > 2$, $G = B_r(K)$ or $D_r(K)$, $r \geq 12$ for $G = B_r(K)$ and $r \geq 14$ for $G = D_r(K)$. Assume that $l(\varphi) \geq p$.*

1. *Suppose that* $\sum_{i=4}^{r} a_i \neq 0$, *or* $a_3 > 1$, *or* $l(\varphi) > p$ *and* $a_2a_3 \neq 0$, *or* $l(\varphi) > p+1$ *and* $a_2 > 2$. *Then for a unipotent element* $x \in G$ *of order* p *the image* $\varphi(x)$ *has at least* $2(r-2)^3$ *Jordan blocks of size* p.

2. *Set* $\Omega = \{(p-1)\omega_1 + \omega_4, (\frac{p-1}{2})\omega_2 + \omega_3, (p-2)\omega_1 + \omega_2 + \omega_3, \omega_1 + (\frac{p+1}{2})\omega_2; a_1\omega_1 + 3\omega_2, a_1 \geq p-3\}$.
If ω *satisfies the assumptions of Item* 1, $\omega \notin \Omega$ *and, furthermore,* $\omega \neq \omega_1 + \omega_4$ *for* $p = 3$, *then* $\varphi(x)$ *has at least* $2(l(\varphi) - p + 2)(r-2)^3$ *such blocks.*

Theorems 1–3 yield lower estimates for the coranks of the images of arbitrary unipotent elements in relevant representations. Set $N(G) = (r-2)^3/8$ for $G = A_r(K)$, $2(r-2)^3$ for $G = B_r(K)$ or $D_r(K)$, and $(r-1)^3$ for $G = C_r(K)$. For all types put $N(G,\varphi) = (l(\varphi) - p + 2)N(G)$.

Corollary 4. *Let a representation* φ *satisfy the assumptions of Theorem* 1, *Theorem* 2, *or Theorem* 3 *for* $G = A_r(K)$, $C_r(K)$, *or* $B_r(K)$ *and* $D_r(K)$, *respectively. Assume that* M *is a* G*-module affording* φ. *Then for a nontrivial unipotent element* $x \in G$ *one has* $\dim(x-1)M \geq (p-1)N(G)$. *Moreover, if the assumptions of Item* 2 *of Theorem* 1 *or Theorem* 3 *hold for* $G = A_r(K)$ *or* $B_r(K)$ *and* $D_r(K)$, *respectively, and* ω *is not one of the exceptional weights mentioned in Theorem* 2 *for* $G = C_r(K)$, *then* $\dim(x-1)M \geq (p-1)N(G,\varphi)$.

Theorems 1 and 2 and the part of Corollary 4 concerning the special linear and symplectic groups were announced in [3].

These results can be easily transferred to irreducible representations of finite classical groups in defining characteristic. They can be useful for recognizing representations and linear groups.

There are examples in [2] showing that the assertions of Theorems 1 and 3 do not hold for arbitrary p-restricted p-large representations. Restrictions on the group rank in Theorems 1–3 are caused by using Lubeck's results [1] on lower estimates for the dimensions of irreducible representations of simple algebraic groups and representations of such groups of small dimensions. If we reduce these restrictions, more exceptions can occur.

II. Restrictions of p-large representations to subsystem subgroups. In what follows a subsystem subgroup is a subgroup of a simple algebraic group generated by the root subgroups associated with all roots of some subsystem of the root system. The restrictions of p-large representations of classical algebraic groups to subsystem subgroups of the maximal rank with two simple components are investigated. Under certain restrictions on the component ranks it is proved that the restriction of a p-restricted irreducible representation of such group in characteristic p to a subsystem subgroup of the form indicated above has a composition factor equivalent to the tensor product of p-large representations of the components if the value of the highest weight on the maximal root is at least $2p$.

Theorem 5. *Let* φ *be a* p*-restricted irreducible representation of* G, $H \subset G$ *be a subsystem subgroup of type* $A_k \times A_{r-k-1}$ *for* $G = A_r(K)$, $B_k \times D_{r-k}$ *for* $G = B_r(K)$, $C_k \times C_{r-k}$ *for* $G = C_r(K)$, *and* $D_k \times D_{r-k}$ *for* $G = D_r(K)$; *and* H_j, $j = 1, 2$, *be the simple components of* H. *Assume that* $l(\varphi) \geq 2p$; $2 \leq k \leq r-3$ *for* $G = A_r(K)$, $3 \leq k \leq r-4$ *for* $G = B_r(K)$, $2 \leq k \leq r-2$ *for* $G = C_r(K)$, *and* $4 \leq k \leq r-4$ *for* $G = D_r(K)$. *Then the restriction* $\varphi|H$ *has a factor of the form* $\varphi_1 \otimes \varphi_2$ *where* φ_j *is a* p*-large representation of* H_j.

The group $A_r(K)$ has an irreducible representation φ with $l(\varphi) = 2p-2$ whose restriction to H has no composition factors mentioned in Theorem 5.

Lemma 6. *Let* $G = A_r(K)$, *the subgroups* H *and* H_j *be such as in Theorem* 5, *and* φ *be an irreducible representation of* G *with highest weight* $(p-1)\omega_1 + (p-1)\omega_r$. *Then the restriction* $\varphi|H$ *has no composition factors of the form* $\varphi_1 \otimes \varphi_2$ *where* φ_j *is a* p*-large representation of* H_j.

These results can be applied for investigating the behaviour of unipotent elements from proper subsystem subgroups in irreducible representations of the classical groups, first of all, elements that have several nontrivial Jordan blocks in the standard realization of the group. They will permit one to obtain new estimates for the number of Jordan blocks of the maximal size in the images of unipotent elements for different classes of elements and representations.

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