# A GEOMETRIC METHOD FOR RATIONAL EXPONENTIAL GROUPS

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Abstract. We will prove that any $\mathbb{Q}$-subgroup of the $\mathbb{Q}$-free group $F_{\mathbb{Q}}(X)$ is $\mathbb{Q}$-free.

## 1. Rational complexes

Through this article, $\mathbb{Q}^+$ is the multiplicative group of positive rational numbers. Any 1-dimensional complex in this article is supposed to be connected. Suppose $C$ is a 1-dimensional complex. For any vertex $v$ the set of all closed pathes with the terminal point $v$ will be denoted by $C(v)$. Suppose $\mathbb{Q}^+$ acts on $C(v)$. For any $p \in C(v)$ and $\alpha \in \mathbb{Q}^+$, we use the notation $p^{(\alpha)}$ for the result of action of $\alpha$ on $p$.

**Definition 1.1.** A simplification operation of $C(v)$ consists of
1- deleting every part of the form $ee^{-1}$ or $e^{-1}e$.
2- replacing a part of the form $p^{(\alpha)}p^{(\beta)}$ by $p^{(\alpha+\beta)}$.

By the operation of the form 2, one can delete any part of the form $p^{(\alpha)}p^{(\beta)}(p^{(\alpha+\beta)})^{-1}$ in the elements of $C(v)$. We say that $p_1, p_2 \in C(v)$ are homotopic, if $p_1$ can be transformed to $p_2$ by means of finitely many simplification operations. In this case we use the notation $p_1 \simeq p_2$. The homotopy class of $p$ is denoted by $[p]$. We say that a path $p \in C(v)$ is reduced if

$$l(p) = \min_{q \in [p]} l(q)$$

where $l(p)$ denotes the length of $p$. It is easy to see that for any natural number $m$, we have $p^{(m)} \simeq p^m$.

**Definition 1.2.** Let $C$ be a 1-dimensional complex with an action of $\mathbb{Q}^+$ on every $C(v)$. Suppose further
1- for any vertex $v$, any $p_1, p_2 \in C(v)$ and any natural number $m$

$$p_1^m \simeq p_2^m \Rightarrow p_1 \simeq p_2$$

2- for any $p \in C(v)$ and $\alpha \in \mathbb{Q}^+$

$$(p^{-1})^{(\alpha)} \simeq (p^{(\alpha)})^{-1}$$

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3- for any two vertices $v$ and $v'$, any path $u$ from $v$ to $v'$, any $p \in C(V)$ and $\alpha \in \mathbb{Q}^+$

$$(upu^{-1})^{(\alpha)} \simeq up^{(\alpha)}u^{-1}.$$

Then we say that $\mathbb{Q}^+$ acts on $C$ or $C$ has a compatible action of $\mathbb{Q}^+$.

**Definition 1.3.** Let $C$ be a 1-dimensional complex with a compatible action of $\mathbb{Q}^+$ and let $T$ be a maximal tree in $C$. The retracted complex $C_T$ is a 1-dimensional complex with the following characterizations:
1- it has only one vertex $v$.

2- for any $e \in C \setminus T$ there is a unique edge $\hat{e} \in C_T$.

3- if $\hat{e_1} = \hat{e_2}$ then $e_1 = e_2$.

4- we have $\widehat{e^{-1}} = \hat{e}^{-1}$.

5- $C_T$ has a compatible action of $\mathbb{Q}^+$: we define $p_T^{(\alpha)} = (p^{(\alpha)})_T$, where if $p = e_1e_2 \dots e_n$, then $p_T = \hat{e}_1\hat{e}_2 \dots \hat{e}_n$.

Note that, if we define the action of $\mathbb{Q}^+$ on $C_T$ as in 5, then actually we get a compatible action of $\mathbb{Q}^+$ on $C_T$ and this will be proved in the main article.

**Definition 1.4.** Let $(C, v)$ be a 1-dimensional complex with a unique vertex $v$ which has a compatible action of $\mathbb{Q}^+$. An edge $e$ is called atomic, if for any irreducible $q \in C(v)$ and any natural number $m$

$$e^m \simeq q \Rightarrow e^m = q.$$

An element $p \in C(v)$ is atomic, if all edges of $p$ are atomic.

If $(C, v)$ is a 1-dimensional complex with a unique vertex $v$ and a compatible action of $\mathbb{Q}^+$, then for any set $A$ of closed pathes, a path of the form $p = p_1p_2 \dots p_n$ with $p_i \in A$ is called an $A$-path. The set of such pathes will be denoted by $\langle A \rangle$. Let $A_0$ be the set of all atomic edges (if there exists any), and define

$$A_{n+1} = \{p^{(\frac{1}{m})} : 2 \leq m,\ p \in \langle A_n \rangle\}.$$

Note that we have $A_0 \subseteq A_1 \subseteq \cdots$.

**Definition 1.5.** Suppose $(C, v)$ is a 1-dimensional complex with a unique vertex $v$ and a compatible action of $\mathbb{Q}^+$. Suppose also
1- $C$ has at least one atomic edge.
2- for any atomic path $p$ and any natural numbers $m$ and $n$, we have

$$p^{(\frac{m}{n})} = (p^m)^{(\frac{1}{n})} = (p^{(\frac{1}{n})})^m$$

3- every path is homotopic to an element of $\cup_{n \geq 0} A_n$.
Then we say that $(C, v)$ is a rational bouquet. An arbitrary 1-dimensional

complex $C$ is a called a rational complex, if it has a compatible action of $\mathbb{Q}^+$ and for any maximal tree $T \subseteq C$, the retract $C_T$ is a rational bouquet.

## 2. Fundamental group

A rational exponential group is a group $G$ such that for any natural number $n$, any element of $G$ has a unique $n$-th root. In this case $\mathbb{Q}^+$ acts of $G$ by

$$g^{\frac{m}{n}} = (g^{\frac{1}{n}})^m$$

and so $G$ is an exponential group in the sense of [7]. As we know, the class of rational exponential groups is variety and so for any set $X$, there exists a free element in this variety which we will denote it by $F_{\mathbb{Q}}(X)$. The main aim of this article is to show that any rational subgroup of $F_{\mathbb{Q}}(X)$ is again free in the variety of rational exponential groups.

**Definition 2.1.** Let $C$ be a rational complex and $v$ be any vertex of $C$. Let

$$\pi_{\mathbb{Q}}(C, v) = \{[p] : \ p \in C(v)\}.$$

Define a binary operation on this set by $[p][q] = [pq]$. It can be shown that $\pi_{\mathbb{Q}}(C, v)$ is a rational exponential group, which we call it the fundamental group of $C$ based on $v$.

Note that since $C$ is connected, so for any two different vertices $v$ and $v'$, we have

$$\pi_{\mathbb{Q}}(C, v) \cong \pi_{\mathbb{Q}}(C, v'),$$

and further this is an isomorphism of rational groups.

**Theorem 2.2.** *The group $\pi_{\mathbb{Q}}(C, v)$ is free in the variety of rational exponential groups.*

**Definition 2.3.** A colored rational complex is a triple $(C, \Phi, v)$ such that $C$ is a rational complex, $v$ is a vertex and $\Phi \subseteq C(v)$. We introduce a new operation of simplification as follows

any part of the form $p \in \Phi$ can be deleted.

Hence we say that two pathes $p_1$ and $p_2$ are color-homotopic, if one can transform $p_1$ to $p_2$ using a finite set of our newly extended list of simplification operations. In this case we write $p_1 \simeq_* p_2$. The colored homotopy class of $p$ is denoted by $[p]_*$. For colored rational complexes, we use the stronger condition

$$p_1^m \simeq_* p_2^m \Rightarrow p_1 \simeq_* p_2,$$

instead of the condition 1 of 1-2.

For a colored rational complex $(C, \Phi, v)$ we define its fundamental group as

$$\pi_{\mathbb{Q}}(C, \Phi, v) = \{[p]_* : \ p \in C(v)\},$$

where the binary operation is $[p]_*[q]_* = [pq]_*$.

**Theorem 2.4.** *For a colored rational complex $(C, \Phi, v)$ the group $\pi_{\mathbb{Q}}(C, \Phi, v)$ is a rational exponential group and conversely every rational exponential group has the form $\pi_{\mathbb{Q}}(C, \Phi, v)$ for some colored rational complex.*

## 3. Free rational groups

Suppose $X$ is an arbitrary set and $F_0 = X \cup X^{-1}$. For $n \geq 0$ we set

$$F_{n+1} = \{w^{\frac{1}{m}} : \ 2 \leq m, \ w \in \langle F_n \rangle\},$$

where $\langle F_n \rangle$ is the subgroup generated by $F_n$ in $F_{\mathbb{Q}}(X)$.

**Proposition 3.1.** *We have $F_{\mathbb{Q}}(X) = \cup_{n \geq 0} F_n$.*

If $w \in F_n \setminus F_{n-1}$, then we say that $w$ has hight $n$ and we denote the hight of $w$ by $h(w)$. Let $E_0 = F_0$ and $E_n = F_n \setminus F_{n-1}$. Suppose $E = \cup_{n \geq 0} E_n$. Note that $E \subsetneq F_{\mathbb{Q}}(X)$. Every element of $E_n$ has the form $u^{\frac{1}{m}}$, where $m \geq 2$ and $u \in \langle F_{n-1} \rangle$ is not a $m$-th power. Every element with this property is called a *pure radical.* So, $E_n$ consists of pure radicals of hight $n$. Note that $E_n$ is closed under the operation of extracting arbitrary roots.

**Definition 3.2.** The reduced form of elements of $F_{\mathbb{Q}}(X)$ is defined by induction:
1- elements of $F_0$ are reduced.
2- a pure radical $u^{\frac{1}{m}}$ is reduced if $u$ is reduced and $h(u) < h(u^{\frac{1}{m}})$.
3- a product $w = w_1 w_2 \dots w_k$ is reduced if any $w_i$ is reduced, $k$ is minimum and for any $i$ we have $h(w_i) < h(w)$.

Every element of $F_{\mathbb{Q}}(X)$ can be expressed as a reduced form and the theorem below shows that this form is unique.

**Theorem 3.3.** *The reduced form of elements of $F_{\mathbb{Q}}(X)$ is unique up to a permutation.*

Let $G$ be a rational exponential group. We know that $G$ is a homomorphic image of a free rational group, so there is a set $X$ and a normal rational subgroup $K$ in $F_{\mathbb{Q}}(X)$ such that

$$G \cong \frac{F_{\mathbb{Q}}(X)}{K}.$$

We define a colored rational complex $C(X, K) = (C, \Phi, v)$ as follows:
1- $C$ has a unique vertex $v$.
2- for any $w \in E$, there exists a unique edge $e(w)$ in $C$.
3- let $p = e(w_1)e(w_2) \dots e(w_k)$ is any path in $C$ and $\alpha \in \mathbb{Q}^+$. Suppose the reduced form of $(w_1 w_2 \dots w_k)^\alpha$ is $w_1' w_2' \dots w_k'$. Then we have

$$p^{(\alpha)} = e(w_1')e(w_2') \dots e(w_k').$$

4- We have

$$\Phi = \{e(w_1)e(w_2) \dots e(w_k) : \ w_1 w_2 \dots w_k \in K\}.$$

**Theorem 3.4.** *$C(X, K)$ is a colored rational bouquet and $\pi_{\mathbb{Q}}(C(X, K)) \cong G$.*

The next theorem shows that the fundamental group of any colored rational complex is isomorphic to the fundamental group of any rational bouquet obtained by a retraction over any maximal tree.

**Theorem 3.5.** *Let $(C, \Phi, v)$ be a colored rational complex and $T$ is a maximal tree. Then*

$$\pi_{\mathbb{Q}}(C, \Phi, v) \cong \pi_{\mathbb{Q}}(C_T, \Phi_T, v),$$

*where $\Phi_T$ is the set of all $p_T$ with $p \in \Phi$.*

## 4. Covering complex

As we saw in the last theorem of the previous section, to study the structure of free rational groups, it is enough to concentrate on the fundamental groups of rational bouquets. So let $(C, v)$ be a rational bouquet. Let $C'$ be an arbitrary rational complex. By $K$ we denote the set of all vertices and edges of $C'$. Suppose a map $f : K \to C(v)$ is given in such a way that
1- for any vertex $v'$, we have $f(v') = v$.
2- if $e'$ is an edge, then $f(e')$ is also an edge.
3- we have

$$f(e'_1 e'_2 \dots e'_k) = f(e'_1) f(e'_2) \dots f(e'_k).$$

4- we have

$$f(p^{(\alpha)}) = (f(p))^{(\alpha)}.$$

5- $f$ is locally one-one, i.e. it is injective on the neighborhood of any vertex.
Then we say that $C'$ is a *covering complex* for $(C, v)$ and $f$ is called a covering map.

**Theorem 4.1.** *Let $f : C' \to (C, v)$ be a covering map. Then for any $v'$, there is an induced embedding*

$$f^* : \pi_{\mathbb{Q}}(C', v') \to \pi_{\mathbb{Q}}(C, v).$$

Now we are ready to give the main theorem of this article.

**Theorem 4.2.** *Let $(C, V)$ be a rational bouquet and $H$ be a rational subgroup of $\pi_{\mathbb{Q}}(C, v)$. Then there exists a covering complex $C'$ for $(C, v)$ such that $\pi_{\mathbb{Q}}(C', v') \cong H$.*

Therefore we obtain a version of the classical Nielsen-Schreier Theorem for rational exponential groups:

**Corollary 4.3.** *Any rational subgroup of a free rational exponential group is again free exponential.*

The present article is a brief description of the outline of our geometric method and complete proofs will be given in a more explained paper.

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