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On control of the prime spectrum of a finite simple group

Let G be a finite group. We name the π-*spectrum* or the *prime spectrum* of G the set $\pi(G)$ of all prime divisors of its order.

We say that the sections (in particular, subgroups) $H_1, \ldots, H_m$ of G *control the prime spectrum of* G (or *control* $\pi(G)$), if

$$\pi(H_1) \cup \ldots \cup \pi(H_m) = \pi(G). \tag{1}$$

We denote by $c(G)$ the minimal from numbers m for which the above situation for some proper subgroups (or homomorphic images of proper subgroups) H_i of G is possible. It is clear that the definition of $c(G)$ does not depend of that arbitrary sections of proper subgroups or only proper subgroups of G are taking as H_i in (1). The parameter $c(G)$ is not defined only when G is the identity group or a group of prime order.

In the author's article [1] (in preparation), some properties of finite simple non-abelian groups connected with the above notions are studied. Situations of specific interest arise when the number m in (1) is small (for example, $m = c(G)$), or moreover when the subgroups $H_1, \ldots, H_m$ are chose of some special "good" structure, convenient for application to given particular investigation.

In [1] for every finite simple non-abelian group G, some the set $\{H_1, \ldots, H_m\}$ of homomorphic images of proper subgroups of G satisfying the condition (1) is indicated. In any case $m \leq 5$ and every of the sections $H_1, \ldots, H_m$ is a simple non-abelian group, Frobenius group or (in only case) dihedral group. Precisely such selection of sections H_i turn out be useful in a certain concrete situation considered in the last section of [1] (see also [2]).

In this work, some corollaries of results of [1] are given. Note that $c(G/\Phi(G) = c(G)$, where $\Phi(G)$ is the Frattini subgroups of G, as $\pi(G/\Phi(G)) = \pi(G)$.

Further, q denotes a prime power.

Theorem 1. *If G is a finite alternating or classical simple group, then $c(G) \leq 2$. In addition, $c(G) = 1$ if G is isomorphic to one of the groups: A_n for non-prime n, groups $PSp_4(q)$, $P\Omega_{4n+1}(q)$ and $P\Omega^+_{4n}(q)$ for all q and groups $PSL_6(2)$, $PSU_3(3)$, $PSU_3(5)$, $PSU_4(2)$, $PSU_4(3)$, $PSU_5(2)$, $PSU_6(2)$, $PSp_6(2)$, $\Omega_7(2)$.*

Theorem 2. *For any finite simple group G of exceptional Lie type it is $c(G) \leq 5$. Moreover,*

$c(G) = 1$ if G is isomorphic to $G_2(3)$ or group Tits ${}^2F_4(2)'$;

$c(G) = 2$ if G is isomorphic to $G_2(q)$ for $q > 3$ or ${}^3D_4(q)$;

$c(G) \leq 2$ if $G \simeq F_4(q)$ ($c(G) = 2$ for $(q, 6) = 1$);

$c(G) = 3$ if G is isomorphic to $Sz(q)$ for $q > 2$ or ${}^2G_2(q)$ for $q > 3$;

$c(G) \leq 3$ if G is isomorphic to $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$;

$c(G) = 4$ if $G \simeq {}^2F_4(q)$ and
$c(G) \leq 5$ if $G \simeq E_8(q)$.

For sporadic simple groups G the question on the size of $c(G)$ is completely decided in the following theorem.

Theorem 3. *Let G be a finite sporadic simple group. Then $c(G) \leq 5$ and namely:*
$c(G) = 1 \Longleftrightarrow G$ is isomorphic to M_{11}, M_{12}, M_{24}, HS, Mc, Co_3 or Co_2;
$c(G) = 2 \Longleftrightarrow G$ is isomorphic to M_{22}, M_{23}, J_2, J_3, He, Suz, Ru, $O'N$, Co_1, Fi_{22}, Fi'_{24} or F_5;
$c(G) = 3 \Longleftrightarrow G$ is isomorphic to J_1, Ly, Fi_{23}, F_3 or F_2;
$c(G) = 4 \Longleftrightarrow G \simeq F_1$;
$c(G) = 5 \Longleftrightarrow G \simeq J_4$.

Generally, if G is a non-simple finite groups then $c(G) \leq 2$.

Proposition 1. *Let G be a finite group and $c(G) \geq 3$. Then $G/\Phi(G)$ is a simple non-abelian group and $c(G) = c(G/\Phi(G))$.*

In fact, if $G \rhd N > \Phi(G)$ then $G = MN$ for a some maximal subgroup M of G and therefore $c(G) \leq 2$.

Thus from Theorems 1–3, the classification of the finite simple groups [3] and Proposition 1 it follows

Proposition 2. $\{m \mid m = c(G)$ *for a finite group* $G\} = \{1, 2, 3, 4, 5\}$.

As it is seen from Propositions 1, 2 in the considered themes the finite simple non-abelian groups participate essentially.

For a finite solvable group G, the upper boundary $c(G) = 2$ is achieved if and only if every maximal subgroup of G is a Hall subgroup of G. Finite groups G of whose all maximal subgroups are Hall subgroups of G are investigated in [4] and [5].

In connection with the above results the following problems may be interesting.

Problem 1. Find $c(G)$ for every finite simple non-abelian group G.

According to Theorem 1, for classical finite simple groups G to solve Problem 1 is the same that to describe such G with $c(G) = 1$.

Problem 2. For a given finite simple non-abelian group G (of a some specific type), find all finite simple non-abelian groups H (also of a some specific type) such that $\pi(G) = \pi(H)$.

Problem 3. For a given set μ of primes, find all finite simple non-abelian groups H (of a some specific type) such that $\pi(G) = \mu$.

For example, the following problem is a special case of Problems 2 and 3.

Problem 4. Find all finite simple non-abelian groups G such that $\pi(G) = \pi(A_p) = \pi(p\,!)$ for a given prime p.

Problem 5. For finite simple groups G with $c(G) = 2$, describe all pairs $\{M, K\}$ of maximal subgroups of G such that $\pi(G) = \pi(M) \cup \pi(K)$.

Similar problem may be formulated also for some groups with $c(G) > 2$.

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