Finite groups with $\mathbb{P}$-subnormal biprimary subgroups

Viktoryia Kniahina
Gomel Engineering Institute of MES of the Republic of Belarus, Gomel, Belarus

We consider finite groups only. Let $\mathbb{P}$ be the set of all prime numbers. A proper subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G if there is a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime number for all $i = 1, 2, \ldots, n$, [1].

In [1–3] are described finite groups with $\mathbb{P}$-subnormal primary subgroups, 2-maximal subgroups and primary cyclic subgroups respectively. In particular, these groups are dispersive, and hence solvable. We develop this subject by considering groups with $\mathbb{P}$-subnormal biprimary dispersive subgroups. The following theorems are proved.

Theorem 1. *Let p be the greatest prime divisor of the order of G. If every biprimary p-closed pd-subgroup of G is $\mathbb{P}$-subnormal in G, then the quotient group $G/O_p(G)$ is p-nilpotent; in particular, G is p-solvable, and $l_p(G) \leq 2$.*

Theorem 2. *Let q be the smallest prime divisor of the order of G. If every biprimary q-nilpotent qd-subgroup of G is $\mathbb{P}$-subnormal in G, then G is solvable, and $l_q(G) \leq 1$.*

Theorem 1 is proved without using the classification of finite simple groups, and in Theorem 2 this classification is used in the proof of solvability of a group.

Example 1. We cannot drop the requirement ≪p is the greatest prime divisor of |G|≫ in the statement of Theorem 1. For instance, consider the simple group $PSL(2, 11)$. All 2-closed biprimary subgroups of even order of this group are isomorphic to the alternating group A_4 of degree 4, which is $\mathbb{P}$-subnormal in $PSL(2, 11)$.

Example 2. Estimate of the p-length of G in Theorem 1 is precise. For instance, all 3-closed biprimary $3d$-subgroups of the group $[E_{3^2}]A_4$ are $\mathbb{P}$-subnormal, and 3-length of this group is equal to 2.

Example 3. A group with $\mathbb{P}$-subnormal q-nilpotent biprimary qd-subgroups may be a simple group for every $q \geq 3$. The example for $q = 3$ is the group $SL(2, 2^n)$ for every odd $n \geq 3$, and for $q \geq 5$ is the group $PSL(2, q)$. Hence we cannot drop the requirement ≪q the smallest prime divisor of $|G|$≫ in the statement of Theorem 2.

References

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