ON THE CLASS OF FINITE GROUPS WITH PRONORMAL HALL SUBGROUPS

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In this talk, all groups are finite. We denote by π a set of primes. π' denotes the set of all primes which do not belong to π. For an integer n, $\pi(n)$ denotes the set of all prime divisors of n. $\pi(G)$ denotes $\pi(|G|)$.

Recall that a group G with $\pi(G) \subseteq \pi$ is called *a π-group.* A subgroup H of a group G is called *π-Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$.

According to [1] we say that a group G *satisfies* $\mathscr{E}_\pi$ if there exists a π-Hall subgroup in G. If a group G satisfying $\mathscr{E}_\pi$ and every two π-Hall subgroups of G are conjugate in G, then we say that G *satisfies* $\mathscr{C}_\pi$. If a group G satisfying $\mathscr{C}_\pi$ and every π-subgroup of G is included in some π-Hall subgroup of G, then we say that G *satisfies* $\mathscr{D}_\pi$. A group satisfying $\mathscr{E}_\pi$ (resp., $\mathscr{C}_\pi$ and $\mathscr{D}_\pi$) is call *an $\mathscr{E}_\pi$-group* (resp., *a $\mathscr{C}_\pi$-group* and *$\mathscr{D}_\pi$-group*). The symbols $\mathscr{E}_\pi$, $\mathscr{C}_\pi$, and $\mathscr{D}_\pi$ denote the classes of all $\mathscr{E}_\pi$-, $\mathscr{C}_\pi$-, and $\mathscr{D}_\pi$-groups, respectively.

Recall that, by definition of P.Hall, a subgroup H of a group G is said to be *pronormal* if the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$.

As a consequence of the Hall theorem, the Hall subgroups of a solvable groups are pronormal, whereas there are examples of non-pronormal Hall subgroups in non-solvable groups.

We say that a group G *satisfies* $\mathscr{P}_\pi$ (is a *$\mathscr{P}_\pi$-group*, belongs to the *class* $\mathscr{P}_\pi$) if $G \in \mathscr{E}_\pi$ and every π-Hall subgroups in G is pronormal.

It was established in [2] that the Hall subgroups in every finite simple group are pronormal. Furthermore, in [3], it was proved that the π-Hall subgroups are pronormal in every $\mathscr{C}_\pi$-groups. Thus

$$\mathscr{C}_\pi \subseteq \mathscr{P}_\pi \subseteq \mathscr{E}_\pi$$

and every simple $\mathscr{E}_\pi$-group belongs to $\mathscr{P}_\pi$.

The following problem is well-known: for which π the inclusion $\mathscr{C}_\pi \subseteq \mathscr{E}_\pi$ is strict? It was established in [4] that $\mathscr{E}_\pi = \mathscr{C}_\pi$ if $2 \notin \pi$ and there are many examples which show that $\mathscr{E}_\pi \neq \mathscr{C}_\pi$ in general.

Also, the following problem is natural: to find the π for which the inclusions $\mathscr{C}_\pi \subseteq \mathscr{P}_\pi$ and $\mathscr{P}_\pi \subseteq \mathscr{E}_\pi$ are strict respectively. Firstly, we prove the following Proposition.

Proposition 1. *For every set π of primes the following statements are equivalent:*

(1) $\mathscr{C}_\pi = \mathscr{E}_\pi$;
(2) $\mathscr{C}_\pi = \mathscr{P}_\pi$;
(3) $\mathscr{P}_\pi = \mathscr{E}_\pi$.

In the theory of classes of finite group and, in particular, in the theory of Hall's conditions $\mathscr{E}_\pi$, $\mathscr{C}_\pi$, and $\mathscr{D}_\pi$, the the following operations play an important role (cf. [5]):

$\mathrm{S}\mathscr{X} = \{G \mid G$ is isomorphic to a subgroup of $H \in \mathscr{X}\}$;

$\mathrm{Q}\mathscr{X} = \{G \mid G$ is epimorphic image of $H \in \mathscr{X}\}$;

TABLE 1. Is it true that $\mathrm{C}\mathscr{X} = \mathscr{X}$, $\mathscr{X} \in \{\mathscr{E}_\pi, \mathscr{C}_\pi, \mathscr{P}_\pi\}$?

C	$\mathscr{E}_\pi$	$\mathscr{C}_\pi$	$\mathscr{P}_\pi$
S	no	no	no
Q	yes	yes	yes
S_n	yes	no	no
R_0	yes	yes	yes
N_0	no	yes	no
D	yes	yes	yes
E	no	yes	no
E_Z	yes	yes	yes
E_Φ	yes	yes	yes

$\mathrm{S}_n\mathscr{X} = \{G \mid G$ is isomorphic to a subnormal subgroup of $H \in \mathscr{X}\}$;
$\mathrm{R}_0\mathscr{X} = \{G \mid \exists N_i \trianglelefteq G\ (i = 1, \dots, m)$ with $G/N_i \in \mathscr{X}$ and $\bigcap_{i=1}^{m} N_i = 1\}$;
$\mathrm{N}_0\mathscr{X} = \{G \mid \exists N_i \trianglelefteq\trianglelefteq G\ (i = 1, \dots, m)$ with $N_i \in \mathscr{X}$ and $G = \langle N_1, \dots, N_m\rangle\}$;
$\mathrm{D}\mathscr{X} = \{G \mid G \simeq H_1 \times \cdots \times H_m$ for some $H_i \in \mathscr{X}\,(i = 1, \dots, m)\}$;
$\mathrm{E}\mathscr{X} = \{G \mid G$ possesses a series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$ with $G_i/G_{i-1} \in \mathscr{X}$ $(i = 1, \dots, m)\}$;
$\mathrm{E}_\mathrm{Z}\mathscr{X} = \{G \mid \exists N \trianglelefteq G$ with $N \le Z_\infty(G)$ and $G/N \in \mathscr{X}\}$;
$\mathrm{E}_\Phi\mathscr{X} = \{G \mid \exists N \trianglelefteq G$ with $N \le \Phi(G)$ and $G/N \in \mathscr{X}\}$;
$\mathrm{P}\mathscr{X} = \{G \mid \forall H \lessdot G\ G/H_G \in \mathscr{X}\}$.

Here $H \trianglelefteq\trianglelefteq G$ means that H is a subnormal subgroup of G, and $H \lessdot G$ means that H is a maximal subgroup of G. For $H \le G$, we denote by H_G the normal subgroup $H_G = \bigcap_{g\in G} H^g$. Furthermore, $Z_\infty(G)$ and $\Phi(G)$ denote the hypercenter and the Frattini subgroup of a group G respectively.

For classes $\mathscr{E}_\pi$, $\mathscr{C}_\pi$, and $\mathscr{D}_\pi$, the problems of closeness under operations defined above was in the focus of attention of many well-known mathematician during the half of century. These problems play a key role in the theory of these classes (for details, cf. the surveys [6, 7]).

We investigate the problem that whether ore not the class $\mathscr{P}_\pi$ is closed under S, Q, S_n, R_0, N_0, D, E, E_Z, E_Φ and P. The results together with the analog for the classes $\mathscr{E}_\pi$ and $\mathscr{C}_\pi$ are collected (with the exception of P) in the table 1.

More precisely, we have the following

Theorem 1. *The following statements hold:*

(*A*) $\mathrm{C}\mathscr{P}_\pi = \mathscr{P}_\pi$ *for every set* π *of primes and* $\mathrm{C} \in \{\mathrm{Q}, \mathrm{R}_0, \mathrm{D}, \mathrm{E}_\mathrm{Z}, \mathrm{E}_\Phi\}$.
(*B*) *If* $\mathrm{C} \in \{\mathrm{S}, \mathrm{S}_n, \mathrm{N}_0, \mathrm{E}\}$, *then* $\mathrm{C}\mathscr{P}_\pi \ne \mathscr{P}_\pi$ *for a set* π *of primes.*
(*C*) *If* $\mathrm{C} \in \{\mathrm{S}_n, \mathrm{E}\}$ *and* $\mathrm{C}\mathscr{P}_\pi = \mathscr{P}_\pi$ *for a set* π *of primes then* $\mathrm{C}\mathscr{E}_\pi = \mathscr{E}_\pi$ *and* $\mathrm{C}\mathscr{C}_\pi = \mathscr{C}_\pi$.

Corollary 1. *Let* $\mathrm{C} \in \{\mathrm{S}, \mathrm{Q}, \mathrm{S}_n, \mathrm{R}_0, \mathrm{N}_0, \mathrm{D}, \mathrm{E}, \mathrm{E}_\mathrm{Z}, \mathrm{E}_\Phi\}$. *Then the following statements are equivalent:*

(1) $\mathrm{C}\mathscr{P}_\pi = \mathscr{P}_\pi$ *for every set* π *of primes;*
(2) $\mathrm{C}\mathscr{E}_\pi = \mathscr{E}_\pi$ *and* $\mathrm{C}\mathscr{C}_\pi = \mathscr{C}_\pi$ *for every set* π *of primes.*

Recall that a *formation* is a $\langle \mathrm{Q}, \mathrm{R}_0\rangle$-closed class $\mathscr{X}$ of finite groups. If, in addition, $\mathscr{X}$ is E_Φ-closed then the formation $\mathscr{X}$ is is said to be *saturated*. A $\langle \mathrm{Q}, \mathrm{P}\rangle$-closed class is called *Schunk class*. Actually, every saturated formation is a Schunk class.

Corollary 2. *For a set π of primes, $\mathscr{P}_\pi$ is a saturated formation.*

Corollary 2. *For a set π of primes, $\mathrm{P}\mathscr{P}_\pi = \mathscr{P}_\pi$. In particular, $\mathscr{P}_\pi$ is a Schunk class.*

One can give examples of sets π such that, for every $\mathrm{C} \in \{\mathrm{S}, \mathrm{S}_n, \mathrm{N}_0, \mathrm{E}\}$ at least one of the classes $\mathscr{E}_\pi$ and $\mathscr{C}_\pi$ is not closed under C (for instance, one can put $\pi = \{2, 3\}$). It follows from Theorem 1, that for this π and every such closure operation C the class $\mathscr{P}_\pi$ is not C-closed. In particular, $\mathscr{P}_\pi$ is not a Fitting class (recall, that if $\langle \mathrm{S}_n, \mathrm{N}_0\rangle\mathscr{X} = \mathscr{X}$ then $\mathscr{X}$ is called *a Fitting class*).

References

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