Locally finite groups with small element orders

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To V.D.Mazurov on occasion of his 70th birthday

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We use variety terms

- A group of *period* n is a group where the identity $x^n = 1$ holds
- While the *exponent* of a group G is the smallest positive integer n such that $x^n = 1$ for every x in G
  In this sense, a group of exponent 6 is also a group of period 12.
- $\omega(G) = \{n \mid G$ has an element of order $n\}$ is a spectrum of a group G
  $\mu(G)$ is a set of maximal with respect to division elements of spectrum of G

The history of the study of periodic groups with prescribed element orders begins with the famous work by W. Burnside 1902.
We shall note an early interest of Victor Mazurov to this problem.
In 1969 he showed that *finite simple group of exponent 30 is isomorphic to A_5* and so (using reduction of Hall and Higman and Kostrikin's results for n=p-prime) solved the restricted Burnside problem «are there only finitely many finite groups with m generators of exponent n, up to isomorphism?» for exponent 30.

Original Burnside problem «Is a group of a given period n locally finite»?
Sergei Ivanovich Adian made and told earlier the history of Burnside problems for big n: if the period n of a group is big enough, then there exist groups of this period which are not locally finite; so the answer is negative.
There are some positive results for small n.

- (W.Burnside 1902) Obvious for $n = 2$, positive solution for $n = 3$
- (I.N.Sanov 1940) Positive for $n = 4$
- (M.Hall 1958) Positive for $n = 6$
- The problem is still open for other small values of n

Some enhancements

- (D.V.Lytkina 2007) $\omega(G) \subseteq \{1, 2, 3, 4\}$ — described structure and simplified the proof
- (M.F. Newman 1984) — shortened essentially Hall's proof, reducing it to some statement (about 3-elements) which he checked with the help of a computer.
- (I. G. Lysenok 1987) — cleared Newman's proof of computer computations.

Theorem S (D.V.Lytkina,V.D.Mazurov,A.S.Mamontov),2012

Let G be some group of period 12 in which the order of the product of every two involutions is distinct from 6, and let H be the subgroup generated by all involutions of G. Then G is locally finite and one of the following holds.

- *G is a 3-group.*
- *H is an extension of a 3-group by a group of order 2, and G is a split extension of a 3-group by a nontrivial subgroup of Q_8 or $SL_2(3)$.*
- *H is an extension of an elementary Abelian 2-group V by a non-Abelian group of order 6, acting faithfully on V and G/H is 3-group.*
- *$G/O_2(G)$ is an extension of a 3-group by an elementary Abelian 2-group and $H \leq O_2(G)$.*

The result generalizes the theorem of Sanov.

- (Shmidt's theorem 1945) If $H \trianglelefteq G$, H and G/H are locally finite, then G is locally finite
- This allows reduction to earlier (known) results
- Coset enumeration (now with the help of computers)
- (Shunkov's theorem 1972) If G contains an involution i with finite $C_G(i)$ then G is locally finite

By Shmidt's theorem we may assume that $G = H$, i.e. is generated by involutions.

Further proof splits into 2 cases:

- If G does not contain a subgroup $R = 3^{1+2} : 2$, then we use Sanov's theorem (the order of every element ≤ 4).
- If G contains R we show that G contains 3-subgroup of index 2.

Theorem(V.D.Mazurov,A.S.Mamontov)

Let G be a group of period 12 in which the order of the product of every two involutions is distinct from 4. Then G is locally finite.

(Baer 1957, Suzuki 1968)

A conjugacy class C of a finite group G generates a nilpotent subgroup if any two elements from C generate a nilpotent subgroup.

Alperin and Lyons in 1971 provided a shorted proof of this result.
There are many generalizations of this theorem on infinite case, for example:

- (Sozutov, 2000): Generalization for binary-finite groups (i.e. when any two elements generate a finite subgroup)
- (Mamontov, 2004): Generalization for groups with maximality condition for nilpotent groups (every ascending chain of nilpotent subgroups stabilizes)

Question 11.11 placed by Borovik into Kourovka notebook:

Question

Is Baer-Suzuki theorem true in periodic groups?

The answer turns out to be positive for groups of period 12.

Let G be a group of period 12 and $a \in G$. If for every $g \in G$ the subgroup $\langle a, a^g \rangle$ is nilpotent, then $\langle a^G \rangle$ is nilpotent

Useful tool for studying groups of period 12.

- G is generated by involutions
- If x, y, z are involutions from G and $(xy)^3 = (yz)^3 = 1$, then $(xz)^3 = 1$
- Consider further graph Γ whose vertices $V(\Gamma)$ are involutions, $x \sim y$ if $(xy)^3 = 1$, and take some connected component Δ.
- For all $b \in H$ and $a \in \Delta$ we have $a^b \in \Delta$. So Δ is full graph whose vertices form a conjugacy class.
- Using analog of Baer-Suzuki theorem we show that it generates locally finite normal subgroup.

- (Bernhard Neumann, 1937) If $\mu(G) = \{2, 3\}$ then G is locally finite. More precisely, such group is an extension of an elementary Abelian p-group V by a cyclic q-group acting freely on V. Here $\{p, q\} = \{2, 3\}$.
- (M.Newman, 1979) Let $\mu(G) = \{2, 5\}$. Then G is either an extension of an elementary Abelian 2-group A by a group P of period 5 acting freely on A, or an extension of an elementary Abelian 5-group by a group of order 2.
- (N.D.Gupta, V.D.Mazurov, 1999) If $\omega(G) \subset \{1, 2, 3, 4, 5\}$, then either G is locally finite, or contains nilpotent normal Sylow subgroup S such that G/S is a 5-group.
- (E.Jabara, 2004) If a group P of period 5 acts freely on an Abelian $2, 3$-group, then $|P| = 5$.
- (Mazurov, 2000) If $\omega(G) = \{1, 2, 3, 4, 5\}$, then either $G \simeq A_6$ or $G = VC$, where V is a nontrivial elementary-Abelian normal 2-subgroup of G and $C \simeq A_5$.

- **Summary**: If $\omega(G) \subseteq \{1, 2, 3, 4, 5\}$ then either G is locally finite, or G is a group of exponent 5.
- (Mazurov,Mamontov, 2009) If $\omega(G) = \{1, 2, 3, 5, 6\}$, then G is locally finite and soluble.
- (Mazurov, 2010) $\omega(G) = \{1, 2, 3, 4, 8\}$
- (Mazurov, Zhurtov, 2011) $2 \in \omega(G) \subseteq \{1, 2, 3, 5, 9, 15\}$.
- (Jabara, Lytkina, Mazurov, 2013) groups of period 36 with involution and without elements of order 6

In this direction I would like to present the following

Theorem, 2013

A group of period 12 *without elements of order 12 (i.e. group with spectrum* $\{1, 2, 3, 4, 6\}$*) is locally finite.*

- Show that $O_2(A_4) \subseteq O_2(G)$ (G.Havas, M.F. Newman, A.C.Niemeyer, C.C.Sims, 1999 found relations of the form $w^6 = 1$ for $B(3,3;6)$, which were useful in the work)
- Using old tricks further assume that G does not have subgroups isomorphic to A_4 and is generated by involutions.
- Show that (as we had before) if x, y, z are involutions from G and $(xy)^3 = (yz)^3 = 1$, then $(xz)^3 = 1$
- Two elements of order 3 generate a subgroup which is a homomorphic image of either 3^{1+2} or $3^4 \rtimes SL_2(3)$.
- If involution $i \in Z(SL_2(3))$ then for every $g \in G$ $(ii^g)^3 = 1$, so (as we did before) $\langle i^G \rangle$ is locally finite.

Let H be a finitely generated subgroup of G.
Then H is generated by a finite set of involutions I.
Take $K = \{i \in I | i \in Z(SL_2(3))\}$ and $L = \langle K^H \rangle$ is locally finite as a finite product of normal locally finite subgroups.
Set $\overline{H} = H/L$. Any 2 elements of order 3 from $\overline{H}/O_2(\overline{H})$ generate a 3-group. So $\overline{H}/O_2(\overline{H})$ is locally finite.

(A.V.Zavarnitsine, 2013) Describes $B_0(2,3;12)$. Order $2^{66}3^7$, derived length 4, $|O_2(B_0)| = 2^{62}$, $O_{2,3}/O_2 \simeq Z_3^6$, $B_0/O_{2,3} \simeq SL_2(3) * Z_4$.

Recognizing groups in the class of all finite groups by spectrum is another interesting research area. A.V.Vasil'ev will tell more about that in his talk.
There are few examples of groups which we know are recognizable in the class of all groups by spectrum. To be specific:

- (A. Kh. Zhurtov, V. D. Mazurov, 1999): $\mu(G) = \{2^m, 2^m - 1, 2^m + 1\}$ if and only if $G \simeq L_2(2^m)$.
- (D.V.Lytkina, A.A.Kuznetsov, 2007): recognized $L_2(7)$.

Theorem (E.Jabara, D.V.Lytkina, A.S.Mamontov, 2013)

Let G be a periodic group with $\omega(G) = \{1, 2, 3, 4, 5, 8\}$. Then $G \simeq M_{10}$.

Here $M_{10} = A_6.2$ is the 3-transitive Mathieu group of degree 10, i.e. a corresponding maximal subgroup of the sporadic simple group M_{11}

- It is enough to show that a group with given spectrum is locally finite
- If $S_3 \leq G$ show $H = (C_3 \times C_3) : C_4 \leq G$
- Take $g \in H$ with $|g| = 4$ and show using coset enumeration that for any involution t centralizing g we should have $t \in H$, i.e. $t \in \langle g \rangle$
- Using corollary of Shunkov's theorem: in an infinite 2-group any finite subgroup is different from its normalizer, we show that this is not possible
- When $S_3 \not\leq G$ use lemma (Neuman+Levi=Mazurov): Let A be a proper subgroup of H. If there are not elements of order 3 in A and any element from $H \backslash A$ has order 3, then A is normal in H and nilpotent of class ≤ 2.

Thank you for your attention!