Some linear methods in the study of almost fixed-point-free automorphisms

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Part 1. Survey on almost fixed-point-free autmorphisms

The more commutativity, the better

Commutator $[a, b] = a^{-1}b^{-1}ab$

$[a, b] = 1 \Leftrightarrow ab = ba$

measures deviation from commutativity.

Generalizations of commutativity are defined by iterating commutators:

a group is nilpotent of class c if it satisfies the law

$[...[[a_1, a_2], a_3], \ldots, a_{c+1}] = 1$.

Solubility of derived length d:

$\delta_1 = [x_1, x_2]$ and $\delta_{k+1} = [\delta_k, \delta_k]$ (in disjoint variables)

a group is soluble of derived length d if it satisfies $\delta_d = 1$

Automorphisms

Let $C_G(\varphi) = \{x \in G \mid \varphi(x) = x\}$ denote
the fixed-point subgroup of an automorphism $\varphi \in \mathrm{Aut}\, G$.

An automorphism φ is fixed-point-free if $C_G(\varphi) = \{1\}$.

Example (of a “good” result)

If a finite group G admits an automorphism $\varphi \in \mathrm{Aut}\, G$ such that $\varphi^2 = 1$ and $C_G(\varphi) = 1$, then G is commutative.

Solubility and nilpotency of groups with fixed-point-free automorphisms

Theorem (Thompson, 1959)

If a finite group G admits a fixed-point-free automorphism of prime order p, then G is nilpotent.

Theorem (CFSG + . . .)

If a finite group G admits a fixed-point-free automorphism, then G is soluble.

Further questions arise: is there a bound for the nilpotency class?

or for the derived length?

Philosophical remark:

Results modulo other parts of mathematics:

- simple or non-soluble groups are often studied modulo soluble groups: for example,
  determine simple composition factors,
  or the quotient $G/S(G)$ by the soluble radical,
  nowadays by using CFSG;
- soluble modulo nilpotent: for example,
  bounding the Fitting height, or p-length,
  by methods of representation theory;
- nilpotent modulo abelian or “centrality”: typically,
  bounding the nilpotency class or derived length,
  often by using Lie ring methods.

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = p$ prime $C_G(\varphi) = 1$
finite	nilpotent Thompson, 1959
+soluble	nilpotent Clifford, 1930s
+nilpotent	class $\leqslant h(p)$ Higman,1957 Kostrikin–Kreknin,1963
Lie ring	same, by same

Almost fixed-point-free automorphisms

Suppose that an automorphism $\varphi \in \operatorname{Aut} G$ is no longer fixed-point-free but has "relatively small", in some sense, fixed-point subgroup $C_G(\varphi)$ (so φ is "almost fixed-point-free").

Then it is natural to expect that G is "almost" as good as in the fixed-point-free case. In other words,

studying finite groups with almost fixed-point-free automorphisms φ means obtaining restrictions on G in terms of φ and $\underline{C_G(\varphi)}$.

Almost fixed-point-free automorphisms

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Classical examples: Brauer–Fowler theorem for finite groups, Shunkov's theorem for periodic groups.

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = p$ prime $C_G(\varphi) = 1$	$\lvert\varphi\rvert = p$ prime $\lvert C_G(\varphi)\rvert = m$
finite	nilpotent Thompson, 1959	$\lvert G/S(G)\rvert \leqslant f(p, m)$ Fong+CFSG, 1976
+soluble	nilpotent Clifford, 1930s	$\lvert G/F(G)\rvert \leqslant f(p, m)$ Hartley+Meixner, Pettet, 1981
+nilpotent	class $\leqslant h(p)$ Higman, 1957 Kostrikin–Kreknin, 1963	$G \geqslant H$, $\lvert G : H\rvert \leqslant f(p, m)$, H nilp. class $\leqslant g(p)$ EKh, 1990
Lie ring	same, by same	same, EKh, 1990; H ideal, Makarenko, 2006

Remarks to the table

Results giving "almost solubility" and "almost nilpotency" of G (or L) when $C_G(\varphi)$ is "small" cannot be obtained by finding a subgroup (or subring) of bounded index on which φ is fixed-point-free.

For almost solubility Hall–Higman–type theorems are applied (in the case of rank, combined with powerful p-groups).

For almost nilpotency of bounded class, quite complicated arguments are used based on "method of graded centralizers" using the Higman–Kreknin–Kostrikin theorem on fixed-point-free case.

Almost regular in the sense of rank

Definition: rank $\mathbf{r}(G)$ of a finite group G is the minimum number r such every subgroup can be generated by r elements (=sectional rank).

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = p$ prime $C_G(\varphi) = 1$	$\lvert\varphi\rvert = p$ prime $\lvert C_G(\varphi)\rvert = m$	$\lvert\varphi\rvert = p$ prime ($p \nmid \lvert G\rvert$ for insol. G) $\mathbf{r}(C_G(\varphi)) = r$ of given rank
finite	nilpotent Thompson, 1959	$\lvert G/S(G)\rvert \leqslant f(p,m)$ Fong+CFSG, 1976	$\mathbf{r}(G/S(G)) \leqslant f(p,r)$ EKh+Mazurov+CFSG, 2006
+soluble	nilpotent Clifford, 1930s	$\lvert G/F(G)\rvert \leqslant f(p,m)$ Hartley+Meixner, Pettet, 1981	$G \geqslant N \geqslant R \geqslant 1$, $\mathbf{r}(G/N)$, $\mathbf{r}(R) \leqslant f(p,r)$, N/R nilpotent EKh+Mazurov, 2006
+nilpotent	class $\leqslant h(p)$ Higman,1957 Kostrikin–Kreknin,1963	$G \geqslant H$, $\lvert G:H\rvert \leqslant f(p,m)$, H nilp. class $\leqslant g(p)$ EKh, 1990	$G \geqslant N$, $\mathbf{r}(G/N) \leqslant f(p,r)$, N nilp. class $\leqslant g(p)$ EKh, 2008
Lie ring	same, by same	same, EKh, 1990; H ideal, Makarenko, 2006	same

Lie ring methods

Lie rings have commutative addition $+$ and bilinear Lie product $[\cdot,\cdot]$ satisfying Jacobi identity $[[a,b]c]+[[b,c]a]+[[c,a]b]=0$.

Lie rings are “more linear” than groups, which makes them often easier to study.

Lie ring method:

hypothesis on a group   hypothesis on a Lie ring

$$\begin{tikzcd} G \arrow[r] & L \arrow[d, Rightarrow, "\text{a Lie ring theorem}"] \\ G & L \arrow[l] \end{tikzcd}$$

result on the group recovered   result on the Lie ring

Various Lie ring methods:

1. For complex and real Lie groups: Baker–Campbell–Hausdorff formula, EXP and LOG functors
2. Mal'cev's correspondence based on Baker–Campbell–Hausdorff formula for torsion-free (locally) nilpotent groups
3. Lazard's correspondence
   (including for p-groups of nilpotency class $< p$)
4. Lie rings associated with uniformly powerful p-groups

   But most "democratic", for any group:

5. Associated Lie ring

Associated Lie Ring

Definition: associated Lie ring $L(G)$

For any group G: $\quad L(G) = \bigoplus_i \gamma_i(G)/\gamma_{i+1}(G)$

(where $\gamma_i(G)$ are terms of the lower central series)

with Lie product for homogeneous elements via group commutators
$[a+\gamma_{i+1},\ b+\gamma_{j+1}]_{\text{Lie ring}} \ := \ [a,b]_{\text{group}} + \gamma_{i+j+1}$

extended to the direct sum by linearity.

Pluses: Always exists.
Nilpotency class of $G =$ nilpotency class of $L(G)$.
Automorphism of G induces an automorphism on $L(G)$

Minuses: Only about $G/\bigcap \gamma_i(G)$, so only for (residually) nilpotent groups.
Even for these, some information may be lost: e. g., derived length may become smaller.

Automorphisms of Lie rings as linear transformations

Suppose that a Lie ring L admits an automorphism φ of finite order n.

We can adjoin a primitive n-th root of unity ω to the ground ring by forming $\hat{L} = L \otimes \mathbb{Z}[\omega]$.

We can define "eigenspaces" $L_i = \{x \in \hat{L} \mid \varphi(x) = \omega^i x\}$.

Then $n\hat{L} \subseteq L_0 + L_1 + \cdots + L_{n-1}$ and the sum is "almost direct" in the sense that if $x_0 + x_1 + \cdots + x_{n-1} = 0$, then $nx_i = 0$ for all i.

Obviously, $[L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod}\,n)}$. ("Almost $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring".)

Theorem (Higman, 1957, Kostrikin–Kreknin, Kreknin, 1963)

If a Lie ring L admits a fixed-point-free automorphism φ of finite order n (such that $C_L(\varphi) = \{0\}$), then L is soluble of derived length $\leqslant k(n)$;

If in addition $n = p$ is a prime, then L is nilpotent of class $\leqslant h(p)$.

(Earlier: Engel–Jacobson–Borel–Mostow for finite-dimensional only and without upper bounds.)

Combinatorial form

The above theorem is essentially equivalent to the following.

Higman, 1957, Kostrikin–Kreknin, Kreknin 1963

If $L = L_0 + L_1 + \cdots + L_{n-1}$ for additive subgroups L_i such that $[L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod}\,n)}$, then $(nL)^{k(n)} \subseteq {}_{\mathrm{id}}\langle L_0\rangle$ (ideal generated by L_0).

If in addition $n = p$ is a prime, then $\gamma_{h(p)+1}(pL) \subseteq {}_{\mathrm{id}}\langle L_0\rangle$.

When $C_L(\varphi) = 0$ we have $L_0 = 0$,
and the main case is when $L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$.

But in other applications (in particular, when a p-automorphism acts on a p-group) this more general form is applied.

Estimates for Kreknin's and Higman's functions

Kreknin's function bounding derived length $k(n) \leqslant 2^n - 2$

Question: is there a linear bound?

Higman's function bounding nilpotency class

$$h(p) \leqslant \frac{(p-1)^{k(p)} - 1}{p-2} \approx p^{2^p}$$

Higman's conjecture: $h(p) = \dfrac{p^2 - 1}{4}$;

Confirmed for $p = 3, 5, 7, 11$

Higman's examples $h(p) \geqslant \dfrac{p^2 - 1}{4}$

Group-theoretic applications of Kreknin's and Higman's theorems

... are immediate for connected simply connected Lie groups with fixed-point-free automorphism of finite order.

For any nilpotent groups:

Corollary (Higman, 1957)

If a (locally) nilpotent group G has an automorphism $\varphi \in \mathrm{Aut}\, G$ of prime order p such that $C_G(\varphi) = 1$, then G is nilpotent of class $\leqslant h(p)$.

Proof: consider $L(G)$ with the induced automorphism:

$C_{L(G)}(\varphi) = 0 \;\Rightarrow\; L(G)$ is nilpotent of class $\leqslant h(p)$ by the Theorem.

Hence so is G. (Some extra care for infinite nilpotent groups: $L(G) = \bigoplus \sqrt{\gamma_i} \big/ \sqrt{\gamma_{i+1}}$.)

(Result is true for any finite group G, nilpotent by Thompson, 1959.)

When it does not work (so far)

Open problem

Does an analogue of Kreknin's theorem hold for nilpotent groups with a fixed-point-free automorphism of arbitrary finite order n?
that is, is derived length $\leqslant f(n)$?

Here $L(G)$ does not work as derived length is not preserved.

So far known only for $|\varphi|$ a prime (Higman–Kreknin–Kostrikin above),

and $|\varphi| = 4$ (Kovács, 1961);
including almost fixed-point-free $|\varphi| = 4$ (EKh–Makarenko, 1996–2006);

(For arbitrary finite groups with a fixed-point-free automorphism everything is already reduced to nilpotent groups:
1) soluble by classification;
2) Fitting height bounded by Hall–Higman–type theorems.)

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = n$ coprime $C_G(\varphi) = 1$
finite	soluble CFSG
+soluble	Fitting height $\leqslant \alpha(n)$ Shult, Gross, Berger
+nilpotent	Is der. length bounded??
Lie algebra	soluble of d.l.$\leqslant k(n)$ Kreknin, 1963

If it works, it works

Theorem (Folklore)

If a locally nilpotent torsion-free group G has an automorphism $\varphi \in \operatorname{Aut} G$ of finite order n such that $C_G(\varphi) = 1$, then G is soluble of derived length $\leqslant 2^n - 2$.

Proof: Embed G into its Mal'cev completion $\hat{G}$ by adjoining all roots of nontrivial elements;

then φ extends to $\hat{G}$ with $C_{\hat{G}}(\varphi) = 1$.

Let L be the Lie algebra over $\mathbb{Q}$ in the Mal'cev correspondence with $\hat{G}$ given by Baker–Campbell–Hausdorff formula.

Then φ can be regarded as an automorphism of L with $C_L(\varphi) = 0$.

By Kreknin, L is soluble of derived length $2^n - 2$;

hence so is $\hat{G}$, and so is G.

Theorem (EKh, 2010)

If a polycyclic group G has an automorphism $\varphi \in \operatorname{Aut} G$ of finite order n with finite fixed-point subgroup, $|C_G(\varphi)| < \infty$, then G has a subgroup of finite index that is soluble of derived length $\leqslant k(n) + 1$.

Proof: by Mal'cev's theorem, G has a characteristic subgroup H of finite index with torsion-free nilpotent derived subgroup $[H, H]$.

Now Folklore's theorem above can be applied to $[H, H]$, so H is soluble of derived length $\leqslant k(n) + 1$.

(Earlier Endimioni 2010 also proved almost nilpotency of class $\leqslant h(p)$ in the case when a polycyclic group G admits an automorphism $\varphi \in \operatorname{Aut} G$ of prime order p with finite fixed-point subgroup, $|C_G(\varphi)| < \infty$.)

Remark: no bounds for the index of those subgroups...

φ $C_G(\varphi)$ G	$\lvert\varphi\rvert = n$ coprime $C_G(\varphi) = 1$	$\lvert\varphi\rvert = n$ coprime $\lvert C_G(\varphi)\rvert = m$	$\lvert\varphi\rvert = n$ coprime $\mathbf{r}(C_G(\varphi)) = r$
finite	soluble CFSG	$\lvert G/S(G)\rvert \leqslant f(n,m)$ Hartley, 1992 +CFSG	$\mathbf{r}(G/S(G)) \leqslant f(n,r)$ EKh+Maz+CFSG, 2006
+soluble	Fitting height $\leqslant \alpha(n)$ Shult, Gross, Berger	$\lvert G/F_{2\alpha(n)+1}(G)\rvert$ $\leqslant f(n,m)$ Turull+ Hartley+Isaacs	$\mathbf{r}(G/F_{4^{\alpha(n)}}(G)) \leqslant f(n,r)$ (Thompson+) EKh+Maz, 2006
+nilpotent	der. length bounded??	??????	??????
Lie algebra	soluble of d.l.$\leqslant k(n)$ Kreknin, 1963	$L \geqslant N$ ideal codim $N \leqslant f(n,m)$ N sol. d.l. $\leqslant g(n)$ EKh+Mak, 2004	same as $\leftarrow$

Non-cyclic groups of automorphisms

Many of the results on bounding the Fitting height of the group G (or certain subgroup) are also valid for any soluble group of automorphisms $A \leqslant \mathrm{Aut}\, G$ of coprime order acting with certain restrictions on $C_G(A)$. First was Thompson's theorem of 1964; many other papers followed, with definitive results by Turull, 1980–90s.

Open questions remain for non-soluble groups of automorphisms. Some progress was made by Turull, Kurzweil.

But some major important problems remain open even for cyclic groups of automorphisms in the non-coprime case.

φ	$\lvert\varphi\rvert = n$ **non-coprime**		
G	$C_G(\varphi) = 1$	$\lvert C_G(\varphi)\rvert = m$	$\mathbf{r}(C_G(\varphi)) = r$
finite	soluble Rowley, 95 +CFSG	$\lvert G/S(G)\rvert \leqslant f(n,m)$ Hartley, 1992 +CFSG	$\mathbf{r}(G/S(G)) \to \infty$ even n prime
+soluble	Fitting height $\leqslant 10 \cdot 2^{\alpha(n)}$ Dade, 1969 $\leqslant \alpha(n)$?? (proved in some cases, Ercan, Güloğlu) polynom. in $\alpha(n)$?? linear in $\alpha(n)$??	Is $\lvert G/F_{f(\alpha(n))}(G)\rvert$ $\leqslant f(n,m)$?? (proved for $\lvert\varphi\rvert = p^k$ Hartley+Turau, 1987) (open even for $\lvert\varphi\rvert = 6$) At least, is Fitting height $\leqslant f(n,m)$??	Bell-Hartley examples for A non-nilp. for A non-cyclic nilp.???
+nilpotent	Is der. length bounded??	??	??
Lie algebra	soluble der. length $\leqslant k(n)$ Kreknin, 1963	ideal codim $\leqslant f(n,m)$ solub. d.l. $\leqslant g(n)$ EKh+Makar. 2004	same as $\leftarrow$

One of open problems for non-coprime case

Kourovka Notebook Problem 13.8(a) (Hartley–Belyaev):

Almost fixed-point-free automorphism of non-coprime order

Suppose that φ is an automorphism of a soluble group G. Is the Fitting height of G bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$?

Equivalent: given an element $g \in G$ in finite soluble group G; is the Fitting height bounded in terms of $|C_G(g)|$?

Bounds (and nice) are known for $|\varphi| = p^k$ being a prime-power (Hartley–Turau), basically because of easy reduction to coprime case.

But even the case $|\varphi| = 6$ is open.

Note that for any finite group G 'generalized Brauer–Fowler theorem' was proved by Hartley + CFSG: the soluble radical has index bounded in terms of $|C_G(g)|$.

Part 2. Using “semisimple” results in “unipotent” situations

“Unipotent” — if a finite p-group P admits an automorphism of order p^n (which cannot be fixed-point-free).

Nevertheless, Kreknin’s theorem was very successfully applied to finite p-groups with an automorphism of order p^k and to pro-p-groups of given coclass in the papers of Alperin, Jaikin-Zapirain, Khukhro, Medvedev, Shalev, Shalev–Zel’manov.

“Unipotent” automorphism of order p

Theorem (Alperin, 1963 – Khukhro, 1985)

If a finite p-group P admits an automorphism φ of prime order p with $|C_P(\varphi)| = p^m$, then P has a subgroup of (p, m)-bounded index that is nilpotent of class $\leqslant h(p) + 1$ (even $\leqslant h(p)$ as noted by Makarenko).

Proofs use associated Lie ring and Higman’s theorem.

“Unipotent” automorphism of order p^n

Theorem (Shalev, 1993 – Khukhro, 1993)

If a finite p-group P admits an automorphism φ of order p^n with $|C_P(\varphi)| = p^m$, then P has a subgroup of (p, m, n)-bounded index that is soluble of p^n-bounded derived length.

Proofs use Kreknin’s theorem.

Shalev’s paper gave “weak” (p, m, n)-bound for the derived length, and EKh’s paper – the final form.

Pro-p-groups of given coclass

Another Lie ring is constructed both in finite p-groups of given co-class, and for pro-p-groups (Shalev, Zel'manov): considering the Lie algebra over transcendental extension $\mathbb{F}_p[\tau]$, with multiplication by τ induced by taking p-th powers in the group.

Theorem (Shalev–Zel'manov, 1992)

Every pro-p group of finite coclass is abelian-by-finite.

Much shorter than earlier proofs (which did not include $p = 2, 3$)

Proof based on Kreknin's theorem applied to a certain Lie algebra.

Alternative direction for unipotent automorphisms

Theorem (Medvedev, 1999)

If a finite p-group P admits an automorphism of prime order p with p^m fixed points, then P has a subgroup of (p, m)-bounded index that is nilpotent of m-bounded class.

Proof uses Higman's theorem.

Theorem (Jaikin-Zapirain, 2000)

If a finite p-group P admits an automorphism of order p^n with p^m fixed points, then P has a subgroup of (p, m, n)-bounded index that is soluble of m-bounded derived length.

Proof uses Kreknin's theorem.

Frobenius groups of automorphisms

(Note there are talks by Makarenko and Ercan.)

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. The condition $C_G(F) = 1$ alone implies that G is soluble of $\alpha(|F|)$-bounded Fitting height.

A new approach is to use the "additional" action of the complement H. By Clifford's theorem every FH-invariant elementary abelian section of G is a free $\mathbb{F}_p H$-module (for various p). Therefore it is natural to expect that properties or G should be close to the corresponding properties of $C_G(H)$ (sometimes depending also on H).

Mazurov's problem

Mazurov's problem 17.72 in Kourovka Notebook

Suppose that both GF and FH are Frobenius groups (so GFH is a "2-Frobenius group").

(a) Is the nilpotency class of G bounded in terms of the class of $C_G(H)$ and $|H|$?

(b) Is the exponent of G bounded in terms of the exponent of $C_G(H)$ and $|H|$?

Part (a) answered in the positive by Makarenko–Shumyatsky, 2010.

Further results (without assuming GF Frobenius) were also obtained about the order, rank, Fitting height, nilpotency class, and exponent of G in terms of $C_G(H)$ and H. Some of these results are easier, some quite difficult, and some problems remain open, like 17.72(b).

Further studies: when kernel almost fixed-point-free; when FH is no longer Frobenius, etc. (talks by Makarenko and Ercan).

Frobenius group of automorphisms with “unipotent” kernel

We saw that results on “semisimple” fixed-point-free automorphisms are applied for studying “unipotent” p-automorphisms of finite p-groups.

Next, “unipotent” application of the following “semisimple” result on metacyclic Frobenius groups of automorphisms.

Theorem (EKh–Makarenko–Shumyatsky, 2011)

Suppose that a finite group G admits a Frobenius groups $FH \leqslant \mathrm{Aut}G$ of automorphisms with cyclic fixed-point-free kernel F. If $C_G(H)$ is nilpotent of class c, then G is nilpotent of $(c, |H|)$-bounded class.

Examples show “cyclic F” is essential. Proof is quite difficult, using graded Lie rings with few non-zero components (after reduction to nilpotent case by CFSG and Clifford’s theorem).

Lie ring result

In fact the following Lie ring result is used for "unipotent" kernel case, in the combinatorial form (roughly speaking).

Theorem (EKh–Makarenko–Shumyatsky, 2011)

Suppose that a Lie ring L admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle\varphi\rangle$ of order n and with complement H of order q such that the fixed-point subring $C_L(H)$ of the complement is nilpotent of class c. Then for (c, q)-bounded numbers $w = w(c, q)$ and $f = f(c, q)$ we have $n^w \gamma_f(L) \leqslant {}_{\rm id}\langle C_L(\varphi)\rangle$.

(Analogy with the combinatorial form of Higman–Kreknin–Kostriikin theorem.)

Frobenius group with “unipotent” kernel: nilpotency class

Theorem (EKh–Makarenko, 2013)

Suppose that a finite p-group P admits a Frobenius group FH of automorphisms with cyclic kernel F of order p^k. Let c be the nilpotency class of the fixed-point subgroup $C_P(H)$ of the complement. Then P has a characteristic subgroup of index bounded in terms of c, $|F|$, and $|C_P(F)|$ whose nilpotency class is bounded in terms of c and $|H|$ only.

Examples shows that the condition of F being cyclic is essential (as it was in the “semisimple” result).

Proof is similar to the proofs of the aforementioned Alperin–Khukhro theorem. The EKh–Makarenkko–Shumyatsky Lie ring theorem takes the role of the Higman–Kreknin–Kostrikin theorem. It is applied to the associated Lie ring of P, and of $\gamma_k(PF)$ for a certain bounded value of k to force required nilpotency of $\gamma_k(PF)$ – and this subgroup has bounded index in P.

Frobenius group with “unipotent” kernel: order, rank, exponent

(Rank is minimum r such that every subgroup is r-generated.)

Theorem (EKh–Makarenko, 2013)

Suppose that a finite p-group P admits a Frobenius group FH of automorphisms with cyclic kernel F of order p^k. Then P has a characteristic subgroup Q of index bounded in terms of $|F|$ and $|C_P(F)|$ such that

(a) *the order of Q is at most $|C_P(H)|^{|H|}$;*

(b) *the rank of Q is at most $r|H|$, where r is the rank of $C_P(H)$;*

(c) *the exponent of Q is at most p^{2e}, where p^e is the exponent of $C_P(H)$.*

The estimates for the order and rank are best-possible,
and for the exponent close to best-possible (and independent of $|FH|$).

The proof uses a reduction to powerful p-groups.

Frobenius group with “unipotent” splitting kernel of prime order

Definition: an automorphism $\varphi \in \operatorname{Aut} G$ is *splitting of prime order p* if

$$\varphi^p = 1 \quad \text{and} \quad xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1 \text{ for all } x \in G.$$

This is equivalent to the following: all elements in the semidirect product $G\langle\varphi\rangle$ outside G are of order p.

Theorem (EKh, 2012)

Suppose that a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel $F = \langle\varphi\rangle$ of prime order p such that φ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$.

(a) *If $C_G(H)$ is soluble of derived length d, then G is nilpotent of (p, d)-bounded class.*

(b) *The exponent of G is bounded in terms of p and the exponent of $C_G(H)$.*

Elimination of operators by nilpotency

Proof of part (a) is based on my method of "elimination of operators by nilpotency" and a result of EKh–Shumyatsky, 1995, (special case of) on groups of prime exponent. When φ acts trivially on G, then $G\langle\varphi\rangle$ is of exponent p, and that theorem applies. Consider relatively free group $F = \langle x_1, x_2, \dots\rangle$ in the variety of nilpotent (of some class) groups of some exponent p^N with operators $\langle\varphi\rangle H$. Scheme of proof:
$[x_1, \dots, x_{c+1}]$ belongs to the normal closure of φ, where c is the bound given by EKh–Shumyatsky theorem. By "Higman's lemma" $[x_1, \dots, x_{c+1}]$ is equal to a product of commutators in φ and $x_1, \dots, x_{c+1}$ involving all these elements, with at least one occurrence of φ. Then each of these commutators is expressed by consequences of the same formula. After substitution into the same formula, $[x_1, \dots, x_{c+1}] =$ similar product of commutators — but now each involving at least <u>two</u> occurrences of φ. And so on, doubling number of occurrences of φ at each step. Since $F\langle\varphi\rangle$ is a finite p-group, it is nilpotent, so in the end that product becomes 1.

Proof of part (b) on exponent

is based on my theorem EKh–1986 giving affirmative solution to an analogue of the Restricted Burnside Problem for groups with a splitting automorphism of prime order p.

We can assume $G = \langle g^{FH} \rangle$, and by EKh-1986 the nilpotency class of G is bounded in terms of p and the number of generators, which is at most $p(p-1)$.

It remains to obtain a bound for the exponent of $G/[G, G]$, which is not difficult.

Automorphisms of p-groups with a partition

Corollary 1

Suppose that a finite p-group P with a partition admits a soluble group of automorphisms A of coprime order such that $C_P(A)$ has derived length d. Then any maximal subgroup of P containing $H_p(P)$ is nilpotent of $(p, d, |A|)$-bounded class.

Note: the nilpotency class of the whole group P cannot be bounded.

Corollary 2

If a finite p-group P with a partition admits a group of automorphisms A that acts faithfully on $P/H_p(P)$, then the exponent of P is bounded in terms of the exponent of $C_P(A)$.

Open questions

Question:

Can similar results be obtained for Frobenius groups of automorphisms with kernel generated by a splitting automorphism of composite (prime-power) order?

Examples show that nilpotency class cannot be bounded (even for cyclic kernel of order p^2 generated by a splitting automorphism and complement of order 2 with abelian fixed points).

Question remains open for the exponent, as well as for the derived length.