Regular subgroups of permutation groups: estimates, algorithms, and applications

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A transitive permutation group is said to be *regular* if the one-point stabilizer of it is trivial. Regular subgroups of permutation groups arise in many natural contexts, for example, in group factorizations [4], Schur rings [7], Cayley graphs [1], etc. Given a group H and a permutation group K, we are interested in the set

$$\mathrm{Reg}(K,H) = \{G \leq K : \ G \text{ is regular and } G \cong H\}.$$

A set $B \subset \mathrm{Reg}(K,H)$ containing exactly one element in each orbit of K acting on $\mathrm{Reg}(K,H)$ by conjugation, is called an H-*base* of K. The cardinality of an H-base of K is denoted by $b_H(K)$.

Using terminology and arguments of [1], one can see that if K is the automorphism group of an object of a concrete category $\mathscr{C}$, then B yields a full set of pairwise non-equivalent representations of this object as a Cayley object over H in $\mathscr{C}$. As $\mathscr{C}$, one can take, for example, the category of finite graphs or other combinatorial structures.

Let $H = C_n$ be a cyclic group of order n. Then, obviously, $b_H(K) \leq c(K)$, where $c(K)$ is the number of the conjugacy classes of full cycles contained in K. It was proved in [5] that the latter number does not exceed n.[1] Thus, in this case $b_H(K) \leq n$. The first main result is given by the following statement proved in [6].

Theorem 1. *A C_n-base of a permutation group of degree n can be constructed in time* poly(n).

Remark. Theorem 1 generalizes a result in [2], where an efficient algorithm for finding a C_n-base of K was constructed in the case where the group K is solvable or 2-closed, i.e., the automorphism group of a graph.

It can be shown that if p is a prime and $H = C_p \times C_p$, then $b_H(K) \leq n$. On the other hand, the following result was recently proved in [3].

Theorem 2. *For every prime p, there exists a p-group $K \leq \mathrm{Sym}(p^3)$ such that $b_H(\Gamma) \geq p^{p-2}$, where $H = C_p \times C_p \times C_p$.*

The groups K in Theorem 2 are not 2-closed. It would be interesting to find 2-closed groups K for which the inequality on Theorem 2 holds true.

References

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[1] More exactly, under the Classification of Finite Simple Groups, $c(K) \leq \varphi(n)$ where φ is the Euler function, ibid.

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