The Exchange Condition for Hypergroups

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Let S be a set, and let μ be a map from $S \times S$ to the power set of S. For any two elements p and q of S, we write pq instead of $\mu(p, q)$ and assume that pq is not empty. For any two non-empty subsets P and Q of S, we define the *complex product* PQ to be the union of the sets pq with $p \in P$ and $q \in Q$. If one of the two factors in a complex product consists of a single element, say s, we write s instead of $\{s\}$ in that product.

Following (and generalizing) Frédéric Marty's terminology [1] we call S a *hypergroup* (with respect to μ) if the following three conditions hold.

1. $\forall\ p,\ q,\ r \in S$: $p(qr) = (pq)r$.
2. $\exists\ e \in S\ \forall\ s \in S$: $se = \{s\}$.
3. $\forall\ s \in S\ \exists\ s^* \in S\ \forall\ p,\ q,\ r \in S$: $p \in q^*r^* \Rightarrow q \in r^*p^*$ and $r \in p^*q^*$.

Each association scheme satisfies the above three conditions with respect to its complex multiplication; cf. [2; Lemma 1.3.1, Lemma 1.3.3(ii), Lemma 1.3.3(i)]. Thus, hypergroups generalize association schemes.

I will explain how association schemes may take advantage of the structure theory of hypergroups. Special attention will be given to the embedding of the theory of buildings as well as the theory of twin buildings into scheme theory; cf. [3].

References

[1] F. Marty, *Sur une généralisation de la notion de groupe*, Huitième Congres des Mathématiciens (1934), 45–59.

[2] P.-H. Zieschang, *Theory of Association Schemes*, Springer Monographs in Mathematics (2005).

[3] P.-H. Zieschang, *Hypergroups*, Max-Planck-Institut für Mathematik Preprint Series **97** (2010).

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