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THE KORTEWEG–DE VRIES REGULARIZATION OF A PARABOLIC EQUATION WITH VARYING TIME DIRECTION

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1. Introduction

Given $Q = (0,1) \times (0,T)$, consider a mixed problem for the generalized Korteweg–de Vries equation

$$Lu = u_t + uu_x + (a(x,t)u_x)_x + \nu u_{xxx} = 0,$$

where $a(x,t)$ is a smooth given function with no restriction on its sign. When $\nu = 0$, we have the parabolic equation which can change the direction of time in an arbitrary smooth manner. Similar equations were studied by V. N. Vragov, A. G. Podgaev [1], S. A. Tersenov [2], N. A. Larkin, V. A. Novikov, N. N. Yanenko [3], A. I. Kozhanov [4]. More references on the subject can be found in the cited papers.

The KdV equation and various generalizations were studied by many authors, see [5–8], on assuming $a(x,t) \le 0$. In other words, the operator

$$u_t + (a(x,t)u_x)_x$$

must be parabolic rather then backward parabolic. It is known that the Cauchy problem for the backward parabolic equation is ill-posed. Our goal here is to show that this fails for the KdV equation.

2. The Mixed Problem

Put

$$W_4 = \{u \in H^4(0,1) \cap H_0^1(0,1);\ u_{xx}(0) = u_x(1) = 0\},$$

$$W_3 = \{u \in H^3(0,1) \cap H_0^1(0,1);\ u_x(1) = 0\},\ \|u(t)\|^2 = \int_0^1 u^2(x,t)dx.$$

Since ν is positive, we let it equal one and consider the following mixed problem

$$Lu = u_t + uu_x + (a(x,t)u_x)_x + u_{xxx} = 0 \text{ in } Q, \tag{2.1}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0,\ t \in (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x),\ x \in (0,1), \tag{2.3}$$

where $a(x,t) \in C^1(\bar{Q}),\ u_0 \in W_4$.

To treat (2.1)–(2.3), we regularize it by the sequence of mixed problems for the Kuramoto–Sivashinsky equation and, passing to the limit on a parameter of regularization, we will arrive at the desired result.

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3. The Regularized Problem

For any $\varepsilon > 0$ we consider the following problem

$$L_\varepsilon u_\varepsilon = u_{\varepsilon t} + u_\varepsilon u_{\varepsilon x} + (au_{\varepsilon x})_x + u_{\varepsilon xxx} + \varepsilon u_{\varepsilon xxxx} = 0, \tag{3.1}$$

$$u_\varepsilon(0,t) = u_{\varepsilon xx}(0,t) = u_\varepsilon(1,t) = u_{\varepsilon x}(1,t) = 0,\ t > 0, \tag{3.2}$$

$$u_\varepsilon(x,0) = u_0(x),\ x \in (0,1). \tag{3.3}$$

Solvability of (3.1)–(3.3) with symmetric boundary conditions

$$u_\varepsilon(0,t) = u_{\varepsilon xx}(0,t) = u_\varepsilon(1,t) = u_{\varepsilon xx}(1,t) = 0$$

was proved in [9]. The Cauchy problem was considered in [5] for $a(x,t) = \varepsilon$ where also were studied asymptotics of solutions to (3.1)–(3.3) as $\varepsilon \to 0$. The difference between this article and [5] is that in our case the limit is singular (one boundary condition must be cancelled) while in [5] the limit is regular.

We construct regular solutions to (3.1)–(3.3) for fixed $\varepsilon > 0$ by the Faedo-Galerkin Method as follows. Let $\{w_j(x)\}$ be eigenfunctions of the following boundary value problem

$$w_{jxxxx} = \lambda_j w_j,\ x \in (0,1),$$
$$w_j(0) = w_{jxx}(0) = w_j(1) = w_{jx}(1) = 0. \tag{3.4}$$

It is known [10] that w_j create a basis for $H^4(0,1)$ which is orthonormal in $L^2(0,1)$. We construct approximate solutions to (3.1)–(3.3) in the standard way

$$u^N(x,t) = \sum_{j=1}^{N} g_{jN}(t) w_j(x),$$

where $g_{jN}(t)$ are solutions to the Cauchy problem for the system of N ordinary differential equations

$$(L_\varepsilon u_\varepsilon, w_j)(t) = (u^N_{\varepsilon t}, w_j)(t) - (1/2)(u_\varepsilon^{N2}, w_{jx})(t) + ((au^N_{\varepsilon x})_x, w_j)(t) -$$
$$-(u^N_{\varepsilon xx}, w_{jx})(t) + \varepsilon(u^N_{\varepsilon xx}, w_{jxx})(t) = 0,$$
$$g_{jN}(0) = (u_0, w_j),\ j = 1, \dots, N. \tag{3.5}$$

By the Carathéodory Theorem, the functions $g_{jN}(t)$ exist on some interval $(0, T_N)$. To extend them to an arbitrary finite interval and to pass to the limit as $N \to \infty$ and $\varepsilon > 0$ fixed, we need a priori estimates. In the calculation to follow we omit the indices ε and N.

Estimate I

Multiplying (3.5) by $g_{jN}(t)$, we come to the equality

$$\frac{d}{dt}\|u(t)\|^2 + u_x^2(0,t) + 2((au_x)_x, u)(t) + 2\varepsilon\|u_{xx}(t)\|^2 = 0. \tag{3.6}$$

Since

$$|\ ((au_x)_x, u)\ | \le A(\|u_{xx}(t)\|\|u(t)\| + \|u_x(t)\|\|u(t)\|),$$

where the positive constant A is defined by $a(x,t)$ and its derivatives, and

$$\|u_x(t)\|^2 \le \|u_{xx}(t)\|\|u(t)\|,$$

we get from (3.6)

$$\frac{d}{dt}\|u(t)\|^2 + \varepsilon\|u_{xx}(t)\|^2 \le C(\varepsilon)\|u(t)\|^2$$

which implies for a.e. $t \in (0,T)$

$$\|u_\varepsilon^N(t)\|^2 + \varepsilon\int_0^t \|u^N_{\varepsilon xx}(s)\|^2 ds \le C(\varepsilon)\|u_0\|^2_{L^2(0,1)}. \tag{3.7}$$

Estimate II

Replacing in (3.5), according to (3.4), w_j by $\lambda_j^{-1}D_x^4 w_j$, we obtain

$$(1/2)\frac{d}{dt}\|u_{xx}(t)\| + (D_x^3u, D_x^4u)(t) + (uu_x, D_x^4u)(t) + ((au_x)_x, D_x^4u)(t)$$
$$+\varepsilon\|D_x^4u(t)\|^2 = 0. \tag{3.8}$$

We estimate separate terms in (3.8) as follows

$$I_1 = |\,(uu_x, D_x^4u)\,| \le \max_{[0,1]}|\,u(x,t)\,|\,\|u_x(t)\|\|D_x^4u(t)\| \le$$

$$\le \frac{\varepsilon}{4}\|D_x^4u(t)\|^2 + \frac{1}{\varepsilon}\|u_x(t)\|^4 \le \frac{\varepsilon}{4}\|D_x^4u(t)\|^2 + \frac{1}{\varepsilon}\|u_{xx}(t)\|^4.$$

By the Ehrling inequality,

$$I_2 = (D_x^3u, D_x^4u)(t) \le \frac{\varepsilon}{4}\|D_x^4u(t)\|^2 + C(\varepsilon)\|D_x^2u(t)\|^2,$$

$$I_3 = ((au_x)_x, D_x^4u)(t) \le \frac{\varepsilon}{4}\|D_x^4u(t)\|^2 + C(\varepsilon)(\|u_{xx}(t)\|^2 + \|u_x(t)\|^2).$$

Inserting $I_1 - I_3$ into (3.8), we come to the inequality

$$\frac{d}{dt}\|u_{xx}(t)\|^2 + \varepsilon\|D_x^4u(t)\|^2 \le C(\varepsilon)(1 + \|u_{xx}(t)\|^2).$$

By the Gronwall inequality and (3.7), we get

$$\|u_{xx}(t)\|^2 + \varepsilon\int_0^t \|D_x^4u(s)\|^2 ds \le C(\varepsilon)\|u_0\|^2_{H^2(0,1)}. \tag{3.9}$$

Estimate III

Differentiating (3.5) by t and replacing w_j by $u^N_{\varepsilon t}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 + ((uu_x)_t, u_t)(t) - (u_{xxt}, u_{xt})(t) + ((au_x)_{xt}, u_t)(t) +$$
$$+\varepsilon(u_{xxt}, u_{xxt})(t) = 0. \tag{3.10}$$

Taking in (3.5) $t = 0$, we get

$$\|u^N_{\varepsilon t}(0)\| \le C\|u_0\|_{W_4}.$$

Using (3.7) and (3.9), we transform (3.10) to the inequality

$$\frac{d}{dt}\|u_t(t)\|^2 + \varepsilon\|u_{xxt}(t)\|^2 \le C(1 + \|u_t(t)\|^2)$$

which implies

$$\|u_t(t)\|^2 + \varepsilon \int_0^t \|u_{xxs}(s)\|^2 ds \le C(\varepsilon)\|u_0\|^2_{W_4}. \tag{3.11}$$

The estimates (3.7), (3.9), (3.11) allow us to extend $u_\varepsilon^N(x,t)$ to any interval $(0,T)$ and to pass to the limit in (3.5) as $N \to \infty$ and $\varepsilon > 0$ fixed. This can be resumed in

Theorem 1. *Let $u_0 \in W_4$, $a \in C^1(\bar{Q})$. Then for every $\varepsilon > 0$ there exists a unique solution to* (3.1)–(3.3) *in the class*

$$u_\varepsilon \in L^\infty(0,T;H^2(0,1)) \cap L^2(0,T;W_4),$$

$$u_{\varepsilon t} \in L^\infty(0,T : L^2(0,1)) \cap L^2(0,T;H^2(0,1) \cap H_0^1(0,1)).$$

Remark 1. From the estimates (3.7), (3.9), (3.11) follows only the existence part of Theorem 1. Uniqueness can be proved by the standard arguments, see [9].

4. The Korteweg–de Vries Equation

To prove solvability of (2.1)–(2.3), we need a priori estimates of u_ε independent of $\varepsilon > 0$.

Estimate IV

Let $\phi(x)$ be a positive smooth function defined on [0,1] with a positive derivative. Multiplying (3.1) by ϕu_ε and integrating over (0,1), we obtain

$$\frac{1}{2}\frac{d}{dt}(\phi, u_t^2)(t) - \frac{1}{3}(\phi_x, u^3)(t) - (au_x, (\phi u)_x)(t) -$$
$$-(u_{xx}, (\phi u)_x)(t) + \varepsilon(u_{xx}, (\phi u)_{xx})(t) = 0. \tag{4.1}$$

We treat the separate terms in (4.1) as follows

$$I_1 = \mid (\phi_x, u^3)(t) \mid = \mid (\frac{\phi_x}{\phi}\phi, u^3)(t) \mid \le C\|u_x(t)\|(\phi, u^2)(t) \le$$

$$\le \eta\|u_x(t)\|^2 + C(\eta)(\phi, u^2)^2(t),\ \eta > 0;$$

$$I_2 = -(au_x, (\phi u)_x)(t) \le 2A(\phi, u_x^2)(t) + C(\phi, u^2)(t),$$

where $A = \max_{\bar{Q}} \mid a(x,t) \mid$ and $C > 0$ does not depend on $\varepsilon > 0$.

$$I_3 = \varepsilon(u_{xx}, (\phi u)_{xx})(t) = \varepsilon(u_{xx}, [\phi u_{xx} + 2\phi_x u_x + \phi_{xx}u])(t) \ge$$

$$\ge \frac{\varepsilon}{2}(\phi, u_{xx}^2)(t) - \varepsilon C(\phi, u_x^2)(t) - \varepsilon C(\phi, u^2)(t)$$

with $C > 0$ independent of $\varepsilon > 0$.

$$I_4 = -(u_{xx}, (\phi u)_x)(t) = \frac{3}{2}(\phi_x, u_x^2)(t) + \frac{1}{2}\phi(0)u_x^2(0).$$

Inserting $I_1 - I_4$ into (4.1), taking into account that ϕ_x is an arbitrary positive on [0,1] function and taking $\eta > 0$ and $\varepsilon > 0$ sufficiently small, we find

$$\frac{d}{dt}(\phi, u^2)(t) + (\phi, u_x^2)(t) + \varepsilon(\phi, u_{xx}^2)(t) \le C(1 + (\phi, u^2)^2(t)), \tag{4.2}$$

where $C > 0$ does not depend on $\varepsilon > 0$.

It is known [11], that the inequality

$$\frac{d}{dt}(\phi, u^2)(t) \le C(1 + (\phi, u^2)^2(t)),\ (\phi, u^2)(0) = (\phi, u_0^2)$$

has a solution on some interval $(0, T_0)$, $T_0 > 0 :\ (\phi, u^2)(t) \le C(\|u_0\|_{L^2(0,1)}, T_0)$.

Returning to (4.2), we find

$$\|u_\varepsilon(t)\|^2 + \int_0^t \|u_{\varepsilon x}(s)\|^2 ds + \varepsilon \int_0^t \|u_{\varepsilon xx}(s)\|^2 ds \le C,\ t \in (0, T_0). \tag{4.3}$$

Estimate V

Differentiating (3.1) by t, multiplying the result by $\phi u_{\varepsilon t}$ and integrating over (0,1), we derive

$$\frac{1}{2}\frac{d}{dt}(\phi, u_t^2)(t) + (\phi(uu_x)_t, u_t)(t) - (u_{txx}, (\phi u_t)_x)(t) -$$
$$-((au_x)_t, (\phi u)_{xt})(t) + \varepsilon(u_{xxt}, (\phi u_t)_{xx})(t) = 0. \tag{4.4}$$

Estimating the separate terms, we obtain

$$I_1 = \frac{1}{2}(\phi(u^2)_{xt}, u_t)(t) = -(\phi(uu_t), u_{xt})(t) - (\phi_x uu_t, u_t)(t) \le$$
$$\le \eta(\phi, u_{xt}^2)(t) + C(\eta)(1 + \|u_x(t)\|^2)(\phi, u_t^2)(t),$$
$$I_2 = -(u_{xxt}, (\phi u_t)_x)(t) = \frac{3}{2}(\phi_x, u_{xt}^2)(t) - \frac{1}{2}(\frac{\phi_{xxx}}{\phi}, \phi, u_t^2)(t) + \frac{1}{2}\phi(0)u_{xt}^2(0),$$
$$I_3 = -((au_x)_t, (\phi u_t)_x))(t) \le C\{(\phi, u_{xt}^2)(t) + (\phi, u_t^2)(t) + (\phi, u_x^2)(t)\},$$
$$I_4 = \varepsilon(u_{xxt}, (\phi u_t)_{xx})(t) \ge \frac{\varepsilon}{2}(\phi, u_{xxt}^2)(t) - \varepsilon C(\phi, u_{xt}^2)(t) - \varepsilon C(\phi, u_t^2)(t)$$

with $C > 0$ independent of $\varepsilon > 0$.

Inserting $I_1 - I_4$ into (4.4) and repeating the arguments exploited by proving (4.2), we come to the inequality

$$\frac{d}{dt}(\phi, u_t^2)(t) + (\phi, u_{xt}^2)(t) \le C(1 + \|u_x(t)\|^2)(\phi, u_t^2)(t).$$

Using (4.3), we obtain

$$(\phi, u_t^2)(t) + \int_0^t (\phi, u_{xs}^2)(s) ds \le C,\ t \in (0, T_0),$$

therefore,

$$\|u_{\varepsilon t}(t)\|^2 + \int_0^t \|u_{\varepsilon xs}(s)\|^2 ds \le C,\ t \in (0, T_0).$$

Solvability of (2.1)–(2.3)

Let $v \in L^2(0, T_0; W_4)$, then (3.1)–(3.3) can be rewritten as

$$\int_0^{T_0} \{(u_{\varepsilon t}, v)(t) + (u_\varepsilon u_{\varepsilon x}, v)(t) + (u_{\varepsilon x}, v_{xx})(t) -$$

$$-(au_{\varepsilon x}, v_x)(t)\}dt + \varepsilon \int_0^{T_0} (u_{\varepsilon xx}, v_{xx})(t)dt = 0. \tag{4.5}$$

In view of (4.3) and (3.7), we pass to the limit as $\varepsilon \to 0$ in (4.5) and obtain

$$\int_0^{T_0} \{(u_t, v)(t) + (uu_x, v)(t) + (u_x, v_{xx})(t) - (au_x, v_x)(t)\}dt = 0.$$

Taking into account the properties of u, we can see that for $a.e.\, t \in (0, T_0)\, u(x,t)$ is a weak solution to the following problem

$$(u_x, v_{xx})(t) - (au_x, v_x)(t) = (F, v)(t), \tag{4.6}$$

where

$$F = -u_t - uu_x \in L^2(0,1).$$

It can be shown also that a weak solution, $u \in H_0^1(0,1)$ of (4.6) is strong, i.e., $u \in W_3$. Summarizing all the facts above, we come to the following assertion.

Theorem 2. *Let $u_0 \in W_4$, $a \in C^1(\bar{Q})$. Then there is $T_0 > 0$ such that the problem* (2.1)–(2.3) *has a unique solution*

$$u \in L^\infty(0, T_0; H_0^1(0,1)) \cap L^2(0, T_0; W_3),$$
$$u_t \in L^\infty(0, T_0; L^2(0,1)) \cap L^2(0, T_0; H_0^1(0,1)).$$

Remark 2. Actually, we have proved only the existence part of Theorem 2, but uniqueness of a strong solution can be proved by the standard arguments.

Remark 3. By density arguments the condition $u_0 \in W_4$ can be weakened to $u_0 \in W_3$.

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