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NONLINEAR EQUATIONS WITH INTEGRALS OF FRACTIONAL ORDER IN WEIGHTED LEBESGUE SPACES

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By Browder – Minty method of monotone operators, existence and uniqueness theorems are proved for three different classes of nonlinear equations involving integrals fractional order in weighted Lebesgue spaces and also norm estimates of solutions are obtained.

Let $\varrho(x)$ be a nonnegative measurable function on the whole real axis $\mathbf{R}$, which is almost everywhere (a. e.) finite and different from zero there. Then $L_p(\varrho)$, $p>1$, is the Banach space of all real-valued measurable functions $u(x)$ on $\mathbf{R}$ with finite norm $\|u\| = \left(\int\limits_{-\infty}^{\infty} \varrho(x)\,|u(x)|^p dx\right)^{1/p}$. We write $u(x)\in L_p^+(\varrho)$ if additionally $u(x)$ is a nonnegative function. For $\varrho(x)=1$ we simple write L_p and $\|\cdot\|_p$, respectively. The dual space to $L_p(\varrho)$ is the space $L_{p'}(\varrho^{1-p'})$ with $p'=p/(p-1)$, the conjugate exponent p, and norm $\|\cdot\|_*$.

Now suppose that function $F(x,t)\,:\,\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ satisfies Caratheodory conditions (i.e., $F(\cdot,t)$ is measurable for all $t\in\mathbf{R}$ and $F(x,\cdot)$ is continuous for almost all $x\in\mathbf{R}$) and let $(Fu)(x)=F[x,u(x)]$ be the corresponding Nemytski operator. Let us write out the sake of reference convenience all the conditions used below on the function $F(x,t)$ determining nonlinearity of the investigated equations. Namely, depending on the class of the investigated equations suppose that $F(x,t)$ satisfies either the conditions (i)–(iii) or (iv)–(vi) ($d_1,\ldots,d_4$ — positive constants):

(i) $|F(x,t)|\le c(x)+d_1\varrho(x)\,|t|^{p-1}$ for a. e. $x\in\mathbf{R}$ and all $t\in\mathbf{R}$ $\left(c(x)\in L_{p'}^+(\varrho^{1-p'})\right)$.

(ii) $(F(x,t_1)-F(x,t_2))\,(t_1-t_2)\ge 0$ for a. e. $x\in\mathbf{R}$ and all $t_1,t_2\in\mathbf{R}$.

(iii) $F(x,t)\cdot t\ge d_2\varrho(x)\,|t|^p-D(x)$ for a. e. $x\in\mathbf{R}$ and all $t\in\mathbf{R}$ $\left(D(x)\in L_1^+\right)$.

(iv) $|F(x,t)|\le g(x)+d_3\left(\varrho^{-1}(x)\,|t|\right)^{1/(p-1)}$ for a. e. $x\in\mathbf{R}$ and all $t\in\mathbf{R}$ $\left(g(x)\in L_p^+(\varrho)\right)$.

(v) $(F(x,t_1)-F(x,t_2))\,(t_1-t_2)>0$ for a. e. $x\in\mathbf{R}$ and all $t_1,t_2\in\mathbf{R}$ such that $t_1\ne t_2$.

(vi) $F(x,t)\cdot t\ge d_4\left(\varrho^{-1}(x)\,|t|\right)^{1/(p-1)}|t|-D(x)$ for a. e. $x\in\mathbf{R}$ and all $t\in\mathbf{R}$ $\left(D(x)\in L_1^+\right)$.

Let use notice that if the conditions (i)–(iii) are fulfilled, then the Nemytski operator $\mathbf{F}$, associated with the function $F(x,t)$, is a bounded and continuous, monotone, coercive mapping from the whole space $L_p(\varrho)$ into $L_{p'}(\varrho^{1-p'})$ and if the conditions (iv)–(vi) are fulfilled, then the operator $\mathbf{F}$ is a bounded and continuous, strictly monotone, coercive mapping from the whole space $L_{p'}(\varrho^{1-p'})$ into $L_p(\varrho)$ (see e. g. [1–3]). The simplest example of a function $F(x,t)$ satisfying the conditions (i)–(iii), (v) is $F(x,t)=\varrho(x)\,t^{p-1}$, where p is an even number.

Let us first consider equation which are simpler for investigation.

Theorem 1. *Let be $p\ge 2$, $0<\alpha<1$ and $\varrho(x)$ satisfy the condition:*

$$\int\limits_{-\infty}^{\infty} [\varrho(x)]^{2/[2-p(1+\alpha)]}dx<\infty\,. \tag{1}$$

If the function $F(x,t)$ satisfies the conditions (i)–(iii), *then the equation*

$$F[x,u(x)]+\frac{1}{\Gamma(\alpha)}\int\limits_{-\infty}^{x}\frac{u(t)\,dt}{(x-t)^{1-\alpha}}=f(x)$$

has a unique solution $u^*(x) \in L_p(\varrho)$ *for any* $f(x) \in L_{p'}(\varrho^{1-p'})$. *Moreover, if additionally* $D(x) = 0$, *then the inequality* $\|u^*\| \le \left(d_2^{-1}\|f\|_*\right)^{1/(p-1)}$ *holds.*

We note that, under the assumptions of Theorem 1, we have $L_p(\varrho) \subset L_{2/(1+\alpha)}$ and $L_{2/(1-\alpha)} \subset L_{p'}(\varrho^{1-p'})$.

We now consider the nonlinear Abel integral equation of second kind.

Theorem 2. *Let be* $1 < p \le 2$, $0 < \alpha < 1$ *and* $\varrho(x)$ *satisfy the condition:*

$$\int_{-\infty}^{\infty} [\varrho(x)]^{2/[2-p(1-\alpha)]} dx < \infty .$$

If the function $F(x,t)$ *satisfies the conditions* (i), (iii) *and* (v), *then the equation*

$$u(x) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{F[t, u(t)]\, dt}{(x-t)^{1-\alpha}} = f(x)$$

has a unique solution $u^*(x) \in L_p(\varrho)$ *for any* $f(x) \in L_p(\varrho)$. *Moreover, if additionally* $c(x) = 0$ *and* $D(x) = 0$, *then the inequality* $\|u^*\| \le d_1\, d_2^{-1}\, \|f\|$ *holds.*

We note that, under the assumptions of Theorem 2, we have $L_{p'}(\varrho^{1-p'}) \subset L_{2/(1+\alpha)}$ and $L_{2/(1-\alpha)} \subset L_p(\varrho)$.

Let us consider corresponding case that a operator of fractional integration enter the equation nonlinearly.

Theorem 3. *Let be* $p \ge 2$, $0 < \alpha < 1$ *and* $\varrho(x)$ *satisfy the condition (1). If the function* $F(x,t)$ *satisfies the conditions* (iv)–(vi), *then the equation*

$$u(x) + F\left[x, \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(t)\, dt}{(x-t)^{1-\alpha}}\right] = f(x)$$

has a unique solution $u^*(x) \in L_p(\varrho)$ *for any* $f(x) \in L_p(\varrho)$. *Moreover, if additionally* $g(x) = 0$ *and* $D(x) = 0$, *then the inequality* $\|u^* - f\| \le d_3\, d_4^{-1}\, \|f\|$ *holds.*

Finally, we point out that similar results like in Theorems 1–3 also hold for corresponding systems of nonlinear equations involving classical Riemann – Liouville operators of fractional integration

$$\left(I_+^\alpha u\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(t)\, dt}{(x-t)^{1-\alpha}}, \quad \left(I_-^\alpha u\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{u(t)\, dt}{(t-x)^{1-\alpha}}, \quad 0 < \alpha < 1 .$$

REFERENCES

1. *Askhabov S. N.* Singular Integral Equations and Convolution Type Equations with Monotonic Nonlinearity. Maikop: MSTU, 2004. 387 p. (in Russian).
2. *Brezis H., Browder F. E.* Some new results about Hammerstein equations // Bull. Amer. Math. Soc. 1974. V. 80, N 3. P. 567–572.
3. *Zabrejko P., Rogosin S.* Nonlinear Abel equation with monotone operators // J. Electrotechn. Math. 1997. N 1. P. 53–65.